How The Hyperuniverse Behaves.

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Introduction.

- We're all acutely aware that **ZFC** fails to settle some very natural combinatorial set-theoretic statements (for example, CH, $\diamondsuit$, SH, PD...).
- This has lead to the genesis of Gödel's Programme:

Gödel's Programme.

Formulate and justify axioms that result in a significant reduction in incompleteness.

- In this talk, I'll examine how one attempt (you may have heard of), namely Friedman's Hyperuniverse Programme, behaves on different ontological pictures.

Introduction.

Aims:

(A) To argue that a natural setting for many of the techniques provided by the Hyperuniverse Programme is in fact the view on which there is one maximal interpretation of set-theoretic discourse.

(B) However, I'll point out that there are some setbacks for one part of the programme on this view.

Strategy.

- Introductory remarks.
- §1 Potentialism and Actualism.
- §2 The Hyperuniverse Programme.
- §3 Challenges for the Hyperuniverse under Potentialism.
- §4 Actualism, class theory, and the Hyperuniverse.
- §5 A problem for the HyperActualist.
- Conclusion.

§1 Potentialism and Actualism.

- Let's suppose that one is in the game of finding well-justified new axioms for set theory (so a Hamkinsian Multiversism or a Fefermanian Indefinite Universism are out).
- Here, we understand set theory through the Iterative Conception of Set; we see sets as given by iterating the power set operation on ω through the ordinals.
- Questions of set theory (and indeed much of its philosophy) can be answered by analysing:
  - i What subsets are formed at additional stages (width).
  - ii How far the stages extend (height).

§1 Potentialism and Actualism.

- There are then a variety of ways we can look at this subject matter:
- *Potentialism in X.* For any universe/interpretation $\mathcal{V}$, we can extend $\mathcal{V}$ in dimension X.
- *Actualism in X.* There is a universe/interpretation $\mathcal{V}$ that cannot be extended in dimension X, and settles the truth value of all sentences in stages present in $\mathcal{V}$.
- There are then $2^2 = 4$ positions available:
  1. Height Potentialism and Width Potentialism: (Skolem, Friedman)
  2. Height Actualism and Width Potentialism: (Steel) (slightly shoehorned)
  3. Height Potentialism and Width Actualism: (Zermelo, Parsons, Tait, Hellman, Linnebo, Isaacson), (note the philosophical differences here!)

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  4. The *aeterni delicti*: Height Actualism and Width Actualism: (Cantor, Gödel, Koellner, and for a while, Barton)

§1 Potentialism and Actualism.

Each view has its own pleasing features and foibles:

- Height Actualism and Width Actualism:
  - Pro: Aesthetically pleasing, easily deals with quantification, provides unified foundation for mathematics, motivation of principles as axioms clear.
  - Con: Proper classes, failure of categoricity, reinterpretation of swathes of set theory.
- Height Potentialism and Width Actualism:
  - Pro: Quasi-categoricity, *prima facie* avoids proper classes.
  - Con: Revenge paradoxes and quantification (problem for any Potentialism), motivating reflection principles, interpreting forcing.
- Height Actualism and Width Potentialism:
  - Pro: Easily interprets forcing, provides arena for study of generic multiverse.
  - Con: Proper classes, motivating axioms that involve width.
- Height Potentialsm and Width Potentialism:
  - Pro: Elegantly accounts for practice of mathematics.
  - Con: Potentially destroys set theory as study of higher infinite.
- **Question.** How do different attempts to execute Gödel's Programme fare on each philosophical conception of set theory?

§2 The Hyperuniverse Programme.

- The Iterative Conception is a lovely conception of set.
- It provides us with a mathematical conception of sets on which it is clear why the paradoxical collections are not sets, and is 'intuitively natural'.
- However, many people feel that it doesn't get us much (plausibly only $L_{\omega+\omega}$).
- **Proposal.** We think that when I take the power set of a given set, I should form as many and varied sets as I possibly can, and should do this for as long as possible. Call this the *Maximal* Iterative Conception.
- **Problem.** Maximality is fickle. *AC versus AD*, Choiceless cardinals *versus AD*, forcing axioms *versus* large cardinals.

§2 The Hyperuniverse Programme.

- **Proposal.** Provide mathematically precise criteria for how the/a universe(s) are generated, giving a quasi-iterative (or at least procedural) intuitive (though possibly mathematically complex) mathematical conception to underwrite the Maximal Iterative Conception of Set.
- This is what the Hyperuniverse Programme attempts to achieve, by analysing various maximality criteria through consideration of the hyperuniverse: the collection of all countable transitive models of ZFC.

§2 The Hyperuniverse Programme.

Two aspects of the Hyperuniverse Programme:

The Justification of Maximality Criteria.

We wish to formulate and justify various maximality criteria as true in our preferential universe(s).

The Reduction to Countability.

This is studied through analysing the collection of all countable transitive models of ZFC, and then results exported back up to $\mathcal{V}$ (a preferential universe).

- Each view is going to behave very differently with respect to different aspects of the programme.

§3 Challenges for the Hyperuniverse under Potentialism.

- The most natural setting for the Hyperuniverse is possibly under Radical Potentialism (Potentialism in both Height and Width).
- This is because many of the maximality criteria depend upon extensions of a particular $\mathcal{V}$.

The Inner Model Hypothesis.

(Friedman, Antos, Honzik, Ternullo) If ϕ holds in an inner model $I^{\mathcal{V}[G]}$ of an outer model $\mathcal{V}[G]$ of $\mathcal{V}$, then ϕ already holds in an inner model I of $\mathcal{V}$.

- Intuitive characterisation: $\mathcal{V}$ has a very high 'density' of inner models, it is already saturated with inner models satisfying any statement that can be realised in a forcing extension.

§3 Challenges for the Hyperuniverse under Potentialism.

Ordinal Maximality.

$\mathcal{V}$ is ordinal maximal iff it has a lengthening $\mathcal{V}'$ such that for all first-order formulas ϕ and subsets $A \subseteq \mathcal{V}$ belonging to $\mathcal{V}'$, if $\phi(A)$ holds in $\mathcal{V}'$ then $\phi(A \cap \mathcal{V}_\alpha)$ holds in $\mathcal{V}_\beta$ for some pair of ordinals $\alpha < \beta$ in $\mathcal{V}$.

- Intuitive characterisation: $\mathcal{V}$ has already realised reflection properties.

§3 Challenges for the Hyperuniverse under Potentialism.

- So, under Radical Potentialism, we can easily analyse maximality criteria as true or false in a particular $\mathcal{V}$.
- Further, she can also reduce this analysis to the countable; the relevant axioms are first-order over the relevant extensions.
- However, there is a rather simplistic challenge. We do want our $\mathcal{V}$ to satisfy these properties, but for any particular $\mathcal{V}$ satisfying our maximality property Ψ, there is a $\mathcal{V}'$ such that $\mathcal{V} \subset \mathcal{V}'$, but $\mathcal{V}' \not\models \Psi$.
- We're forced to say that $\mathcal{V}$ is preferable to $\mathcal{V}'$, *despite* the fact that $\mathcal{V}'$ contains more sets.
- We might think that this is a feature rather than a bug (and necessitates an unorthdox analysis of set-theoretic truth), but we also might wonder if we can get a bit more 'oomph!' into our justification relative to truth.

§3 Challenges for the Hyperuniverse under Potentialism.

- Case study: Suppose you want to justify the 'truth' of some switch ϕ (e.g. CH).
- Even if any $\mathcal{V}$ satisfying some maximality criteria Ψ is such that $\mathcal{V} \models \phi$, there is always a $\mathcal{V} \subset \mathcal{V}' \models \neg\phi$.
- The more homogeneity we have between the different $\mathcal{V}$s, the stronger our justificatory case. For example, if every $\mathcal{V}$ has the same few levels above V_ω, we will settle many classic independent questions (e.g. problematic statements of second and third-order arithmetic).
- **Remark.** Height Actualism and Width Potentialism performs poorly with respect to the Hyperuniverse Programme, incurring both (i) the weakness of justificatory strength, and (ii) problems with interpreting height extensions.

§3 Challenges for the Hyperuniverse under Potentialism.

- That leaves (from our Potentialist frameworks) Height Potentialism with Width Actualism.
- Here, we can argue that if every $\mathcal{V}$ we countenance contains V_α, claims concerning sets in V_α are bivalent.
- This then lends the picture some justificatory clout.
- But what about width extensions?

§3 Challenges for the Hyperuniverse under Potentialism.

- Using techniques in [Barwise, 1975] and updated in [Antos et al., 2015] we can interpret discourse involving forcing extensions using additional height ($\mathcal{V}$-logic).
- We add the following to the language of ZFC:
  1. A constant symbol $\bar{a}$ for every set a in $\mathcal{V}$.
  2. A constant (predicate) symbol $\bar{\mathcal{V}}$ to denote $\mathcal{V}$.
- And the following rules of inference:
  1. From $\phi(\bar{b})$ for all $b \in a$ infer $\forall x \in \bar{a}\phi(x)$.
  2. From $\phi(\bar{a})$ for all $a \in \mathcal{V}$ infer $\forall x \in \bar{\mathcal{V}}\phi(x)$.

§3 Challenges for the Hyperuniverse under Potentialism.

1. From $\phi(\bar{b})$ for all $b \in a$ infer $\forall x \in \bar{a}\phi(x)$.
2. From $\phi(\bar{a})$ for all $a \in \mathcal{V}$ infer $\forall x \in \bar{\mathcal{V}}\phi(x)$.

- We can then, for interpreting forcing, add another constant symbol $\bar{\mathcal{W}}$ the following axiom:
- $\bar{\mathcal{W}}$ is a transitive model of ZFC containing $\bar{\mathcal{V}}$ as a subset and with the same ordinals as $\bar{\mathcal{V}}$.
- The consequences of a forcing extension can be formulated as the *consistency* in $\mathcal{V}$-logic of these theories.
- The rules and axioms above facilitate class-sized proof codes (as well-founded trees).
- The proof codes appear in $(\mathcal{V})^+$, the least admissible set (model of KP) containing $\mathcal{V}$.
- Thus we can still reduce to the hyperuniverse, as the relevant tools are first-order over $\mathcal{V}^+$

§3 Challenges for the Hyperuniverse under Potentialism.

- The problem of maximality and conceptions of set might resurface again if we want to include reflection properties.
- There is no guarantee if $\mathcal{V}$ satisfies a reflection property, then a $\mathcal{V}'$ (of which $\mathcal{V}$ is an initial segment) will.
- Does $\mathcal{V}'$ have less claim to be maximal than $\mathcal{V}$?

§4 Actualism, class theory, and the Hyperuniverse.

- What would be really nice is if we could just hold our maximality principles to be true, without the need to preferentially select universes on slightly *ad hoc* grounds.
- Actualism provides a natural arena to do this, if ϕ is true in or of V, then it is just plain true *simpliciter*.
- **Problem.** We can't build $(V)^+$.

§4 Actualism, class theory, and the Hyperuniverse.

- Using the plural paraphrase strategy of [Boolos, 1984] and subsequently taken up in [Uzquiano, 2003], we can justify *some* class theory stronger than *NBG*.

"The rocks rained down."
"The set theorists stand in a circle."
"There are some gunslingers such that each has shot the right foot of one of the others."
"There are some (natural) numbers mm such that if n is one of them, then there is a number $n' = (n-1)$ that is also one of the mm."
"There are some sets xx that do not (collectively) form a set, and any set x is one of them iff $x \notin x$."

§4 Actualism, class theory, and the Hyperuniverse.

First philosophical reasons to think non-definable class talk is okay:

> *"There are some sets such that they do not (collectively) form a set and are also not all and only the satisfiers of some first-order formula."*

- For the Actualist, this looks true.
- This is especially so given that the Actualist thinks that there are non-definable classes over most V_κ.

§4 Actualism, class theory, and the Hyperuniverse.

$$\exists C \forall x (x \eta C \leftrightarrow \phi(x))$$

"There are some things cc such that for any x, x is one of them iff $\phi(x)$."

Why should we prohibit unbounded class quantifiers in ϕ given the plural interpretation?

§4 Actualism, class theory, and the Hyperuniverse.

Second philosophical reasons to accept some non-definable class talk:

- Having all classes be first-order definable trivialises Kunen's theorem that $\neg\exists j : V \longrightarrow V$.
- The view also prohibits a $j : \mathfrak{M} \longrightarrow V$ where $On \subset \mathfrak{M}$.

Vickers and Welch

(Vickers-Welch) Suppose $I \subseteq Ord$ witnesses Ord is Ramsey. Then, definably over $\langle V, \in, I\rangle$, there is a transitive class $\mathcal{M}$, and an elementary embedding $\mathbf{j}$: $\langle \mathcal{M}, \in\rangle \longrightarrow \langle V, \in\rangle$ with a critical point.

§4 Actualism, class theory, and the Hyperuniverse.

- **Fact.** You can code up $(V)^+$ in MK.
- So the Universist can interpret forcing over V using V-logic.
- This is more natural than it may seem. Remember all this is heavily coded:
  - By the Bernays generalisation of Cantor's Theorem (no mapping from V onto 'all subclasses' of V, provable in NBG), she should already accept that we can code up claims that 'go beyond' V.
  - **Ordinal maximality:** Any reflection property I can code up in a really strong logic (V-logic) is already realised (V is really maximal!)
  - **Inner Model Hypothesis:** V has lots of inner models, I can't even code up a way that it could have more!

§5 A problem for the HyperActualist.

- So, we've seen that there is at least the *possibility* of the Actualist making use of *maximality criteria*.
- But there's a problem. If we have MK over V with the needed interpretation of the variables, then we don't have the Downward Löwenheim-Skolem property for this theory.
- We can't reduce to the hyperuniverse.
- But is this aspect of the proposal really necessary as long as we have V-logic?

Conclusions.

- I've stated all this very provocatively, as though the Actualist is in a better position than any form of Potentialism to make the case for maximality criteria.
- Really though, I just want to make the space for the Actualist to use forcing arguments where 'V' denotes V.
- **Main Upshot**: An awful lot more than one realises can be coded up (with some pretty beefy logic expansion), and we should be mindful of this when arguing about what resources are legitimate on a particular philosophical view.

Gratias agere.

Thanks for listening! Discussion!

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