A Height-Potentialist View of Set Theory

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1 The Modal-Structural Framework

Background logic: S5 quantified modal logic with second-order or plurals logic, without Barcan or Converse Barcan axioms.

Start with categorical second-order axioms for a given branch of mathematics, e.g. Dedekind-Peano for arithmetic, Dedekind's or Cantor's for the reals, ZF^{2}C for set-theoretic "standard models", i.e. with full power sets, and strongly inaccessible ordinal height, etc. In each case, provide two things:

A "hypothetical component", a translation, S$_{MS}$ of sentences, S, in the language of such a theory, in the case at hand, ZF^{2}C, for S of bounded rank, say ρ:

$$\Box\forall X, R[(\text{“}X \text{ has all ‘sets’ of rank } \leq \rho\text{”} \wedge \wedge Ax) \rightarrow S]^{X}_{\in/R} \qquad (\mathrm{S}_{MS})$$

where $\wedge Ax$ is the conjunction of the ZF^{2}C axioms, the superscript X indicates relativization of all quantifiers to the domain X, and the subscript indicates replacement of '$\in$' with relation variable 'R'. (This in effect expresses that S would hold in any sufficiently high model of the axioms that, logico-mathematically, there might be.)

As a consequence of Zermelo's proof (1930) of the quasi-categoricity theorem and the next step, this translation scheme is bivalent for sentences with bounded quantifiers. In particular, in addition to virtually all of "ordinary mathematics", this includes CH, whether or not we ever settle it.

Next there is a "categorical component", in the form of a modal-existence postulate:

$$\Diamond \exists X, R[\wedge Ax \wedge S]^{X}_{\in/R}.$$

As a result, our translations are not vacuously true conditionals, so that MS is not a form of "if-then'ism" or "deductivism" (once proposed by Russell).

Remark: The elimination of '$\in$' in favor of a quantifable relation variable frees us from positing *sets* as absolute objects, and comports with a structural understanding: any objects whatever standing in the relations specified by the axioms form a model. Furthermore, this avoids the objection that modal language, like tensed language, really makes no sense in connection with pure sets. Note, however, that we are not committed to taking the objects of models as "concrete", especially in the sense of spatio-temporal, as iterated power sets in the transfinite transcend anything recognizable as a spacetime manifold. Rather than "modal nominalism", MS expresses a "modal neutralism".

If S contains unbounded quantifiers, MS takes over a translation pattern due to Putnam. Let S be in prenex form, with $Q_1 x_1, \ldots, Q_n x_n$ the initial unbounded quantifiers. The translation replaces $Q_1 x_1$ with '$Q_1 M_1 Q_1 x_1$ in M_1', where M_1 is a variable over ZF^2C models (X, R satisfying the axioms, as above); for $n > 1$, if Q_n is $\forall$, the translation replaces $\forall x_n$ with $\forall M_n, x_n(M_{n-1} \prec_\in M_n \wedge x_n$ in $M_n \rightarrow \ldots)$; if Q_n is $\exists$, the translate is the same except with $\rightarrow$ replaced by $\wedge$. Here $\prec_\in$ stands for the converse of (proper) end-extension. (If a variable x_i is second-order, then 'in' means '$\subseteq$'. The clauses for the sentential connectives are standard.)

Remark: For first-order sentences, it can be shown that this "Putnam-extendability translation" is faithful, in the sense that the modal theory proves S_{MS} just in case ZFC with the axiom of inaccessibles proves S. (Roberts, Hellman)

The next MS postulate defines a height-potentialist view: "Necessarily any model has a possible extension to a more inclusive one, in particular to a proper end-extension". Letting M, M',etc. range over standard models of the form $< X, R >$, this *Extendability Principle* takes the abbreviated form:

$$\Box \forall M \Diamond \exists M' [M \prec_{\in} M']. \qquad \text{(EP)}$$

This is a modal version of Zermelo's (1930) postulate, independently put forward in modal form (but applied to standard models of Zermelo set theory) by Putnam (1967).

Remark 1: One effect of the EP is that the Putnam translate of some second-order sentences can have the opposite truth-value of the original, to wit, "For every class, some set is co-extensive, or equinumerous, with it": on the single, fixed universe view, these are of course false, but their Putnam-translates are true. (In the next section, this phenomenon is turned to advantage.)

Remark 2: Some philosophers have quibbled with this on account of the possibility of "metaphysically shy" objects, ones that somehow by their nature cannot coexist with more set-like objects than those of a given model (something that wouldn't arise were we, like Parsons, to take our objects always to be genuine *sets*, despite description with modal operators). To handle such objections, the EP can be framed "up to isomorphism", which suffices for mathematics.

Remark 3: More important, what then blocks a revenge argument that a modal version of *inextendable* "proper classes" arises by considering the possibility of the "union" of "all possible models", violating the EP? The answer is that, with the modal operators properly understood along "actualist" lines, purported reference to such monstrosities makes no sense. Just as one cannot "collect" objects that merely might have existed, one could never be in a position to "collect" (even by merely conceiving) "all possisble objects of domains of models that there then merely *might* have been". Harking back to Kripke's understanding of modal discourse (as opposed to David Lewis'), "possible worlds" are merely a *façon à parler.* Officially, we do not quatify over such; rather we use modal operators as primitive.

Remark 4: Note that this is a major advantage of working in a modal language. Zermelo's original non-modal EP *is* subject to revenge: his (1930) is naturally formalized in 2d-order logic; but 2d-order logical comprehension immediately yields the class of all ordinals, of all sets, all models, etc.

2 Resolving the Paradoxes

This is one of MS's strong suits. We illustrate briefly with the Burali-Forti.

- Standard set-theory's resolutions (for ZFC or NBG) are fine for mathematical purposes, but less than satisfying philosophically, as they fail to provide ordinals representing perfectly good well-order relations (e.g. $\in$ itself restricted to the von Neumann ordinals).

- We should strive to meet the **condition** that *any well-ordering relation be represented by a unique ordinal* in the sense that the pairs of the given well-order relation should be in one-one order-preserving correspondence with the pairs of ordinals strictly less than the representing ordinal.

- Elementary observation about ordinals in their natural ordering, $<$ (For simplicity, work with von Neumann ordinals, $\in$-well-orderings with the null set representing the null well-ordering.): Let $\beta\beta$ (plural variable) be downward closed ordinals, i.e. if $\gamma < \delta$ and $\delta \ \eta \ \beta\beta$, then $\gamma \ \eta \ \beta\beta$. (Notation: read '$\eta$' as 'is among' or 'is one of'.) Then we have:

Proposition: *If the $\beta\beta$ are represented by ordinal λ, then λ is the least strict upper bound of the $\beta\beta$; in particular,* $\neg(\lambda \ \eta \ \beta\beta)$.(HW)

- MS can meet the above condition, framed modally: it takes over Zermelo's point that the set/proper class distinction is always *relative to a model.* So a proper-class well-ordering lacking an ordinal rep. in a given model, gains one in all proper extensions. These give rise to new unrepresented wo's, but they are represent*able*, gaining ordinal rep's in yet higher models, and on and on...

- Clearly a similar story can be told for the other paradoxes. (HW exercise.)

3 Climbing Higher: Two Routes to Small Large Cardinals

First Route: Use behavior of model extensions to motivate new axioms. E.g., for strongly inaccessibles, we have

$$\Box\forall M, \alpha \text{ in } M \Diamond\exists M', \kappa \text{ in } M'[\text{Inac}(\kappa) \wedge \kappa > \alpha],$$

since the height of any M is inaccessible and, by EP, that occurs as an individual in a proper extension of M. This motivates adding the Axiom of Inaccessibles, $\forall\alpha\exists\kappa[\text{Inac}(\kappa) \wedge \kappa > \alpha]$, to the ZF^2C axioms and asserting the possibility of models of that theory. The height of such will then be hyperinaccessible, and then the same argument pattern can be repeated to obtain hyper-hyperinaccessibles, and so on. Mahlo cardinals can be obtained by adding to the non-modal set theory the axiom that every normal function on ordinals has an inaccessible fixed point.

Remark 1: As is well know, such results depend on second-order formulations, e.g. of Replacement for strongly inaccessibles, of the fixed-point axiom just mentioned for Mahlos, etc. The arguments break down for first-order theories.

Remark 2: How far this route extends is not entirely clear. (Perhaps precisifying the method would yield a definite limit.) But even if it only takes us through the Mahlos, that is worth something, if only to reinforce the point that adding small large cardinal axioms is well motivated and a natural outgrowth of ZF^2C.

Second Route: If we could motivate adding the second-order reflection scheme (Bernays') (call it R^2) as axioms to, say, Zermelo set theory with second-order Separation but without the Axiom of Infinity, then we would obtain a great unification yielding both Infinity and Replacement^2, and small large cardinals through at least the indescribables. As the story is usually told, one appeals to the universe V of *all sets* as too large to be described by formulas in the language of set theory: any such condition, if it holds (in V) also holds at some initial segment, V_α, with second-order parameters restricted to that segment (via intersections with V_α). This, however, is problematic for the height potentialist who forswears recognizing any absolute universe.s

Resolution: Just as we can motivate Repacement^2 (as possible along with the axioms of Z^2C) by appealing to its mathematical fruitfulness and to *our interests* in investigating ordinally rich structures without thereby appealing to an absolute universe, so we can motivate R^2, both by its unifying power and by citing our interests in studying structures large enough to be indescribable in the relevant sense. It is by no means necessary to admit exceptions to the EP. While we have no guarantee that our interests are reflected in the possibilities of large structures, we can at least adopt such axioms as working hypotheses reflecting the coherence of our concepts, and perhaps that is good enough.