Nonstandard Analysis as a computational foundation

Sam Sanders

SOTFOMIII, Vienna, Sept. 2015

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In this talk, we show that Nonstandard Analysis provides ZFC with a 'computational' foundation.

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We show that Nonstandard Analysis provides a similarly constructive interpretation of mathematics. (Bishop and Connes)

A little test. . .

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$$(\forall x, y \in [0,1])[x \approx y \to f(x) \approx f(y)]$$

$$\downarrow$$

$$(\forall \pi, \pi' \in P([0,1]))(\|\pi\|, \|\pi'\| \approx 0 \to S_\pi(f) \approx S_{\pi'}(f))\big],$$

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OR

$$(\forall k^0)(\forall x, y \in [0,1])(|x - y| < \tfrac{1}{g(k)} \to |f(x) - f(y)| < \tfrac{1}{k})$$

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$$(\forall k')(\forall \pi, \pi' \in P([0,1]))\big(\|\pi\|, \|\pi'\| < \tfrac{1}{s(g,k')} \to |S_\pi(f) - S_{\pi'}(f)| \le \tfrac{1}{k'}\big)$$

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As we will see: the first one! (up to finitistic manipulation)

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Motivation: Many authors have observed the 'constructive nature' of the practice of NSA. (Horst Osswald's local constructivity)

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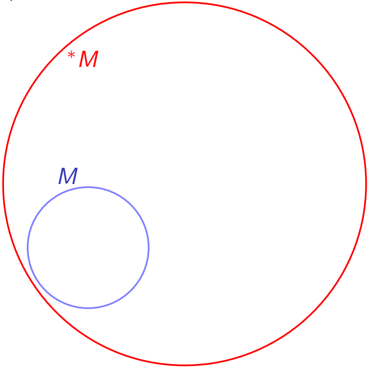

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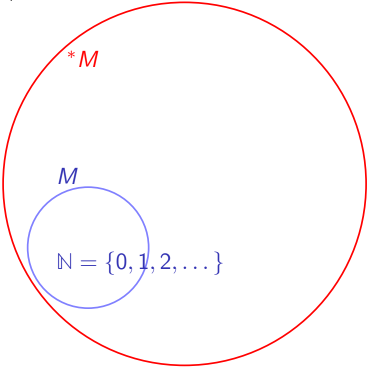

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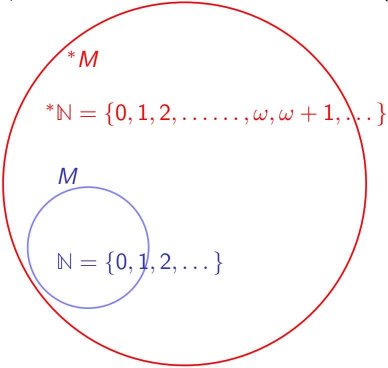

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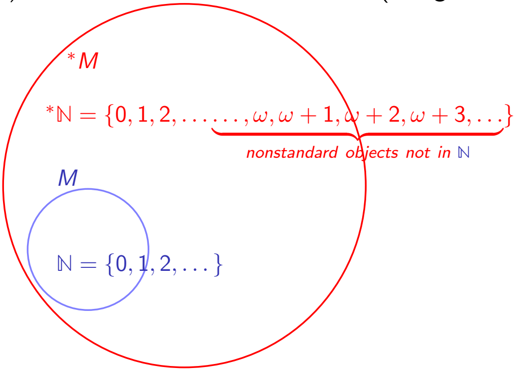

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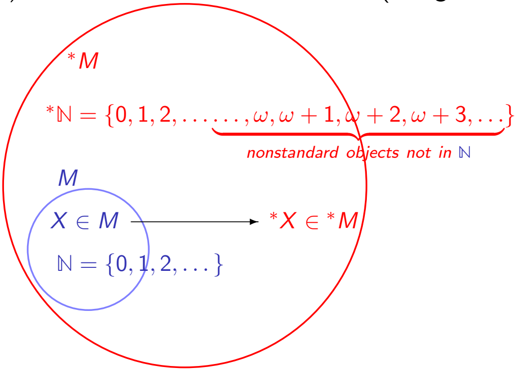

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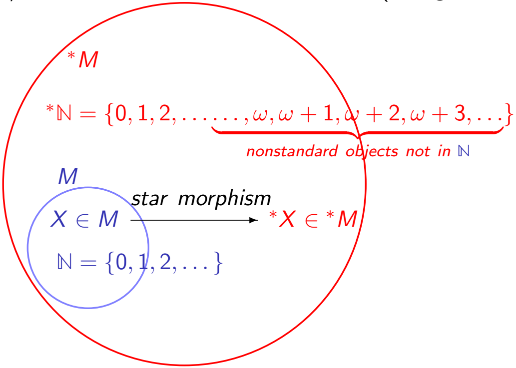

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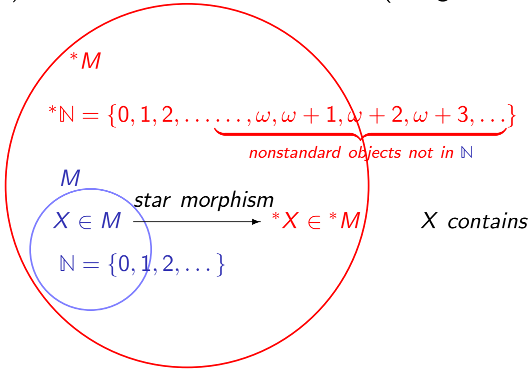

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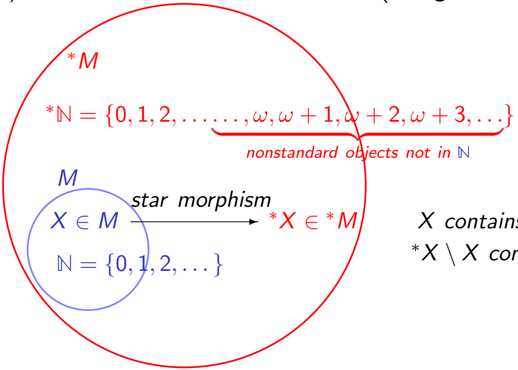

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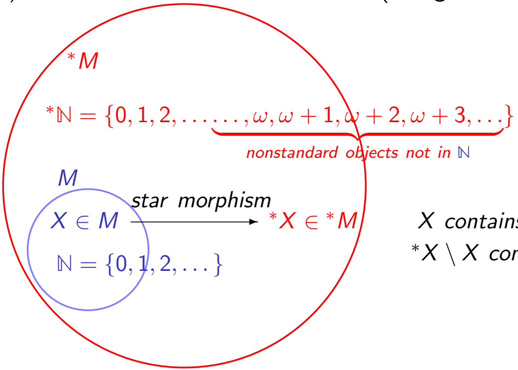

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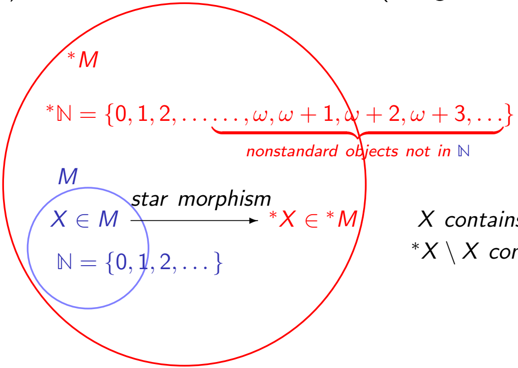

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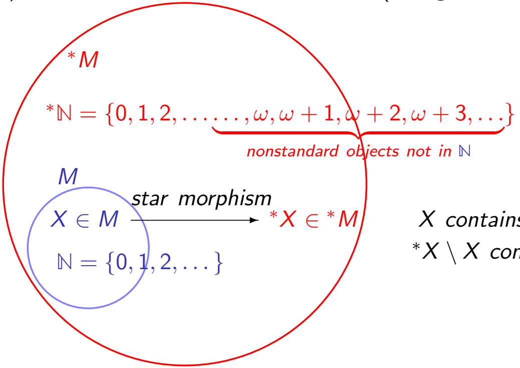

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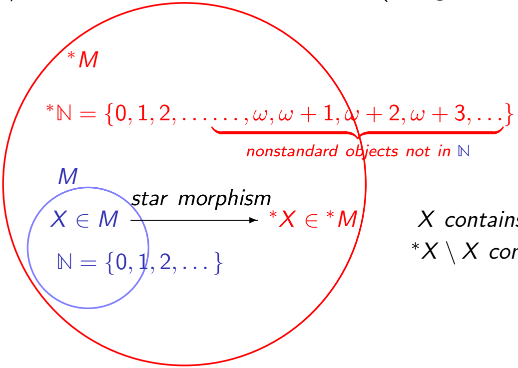

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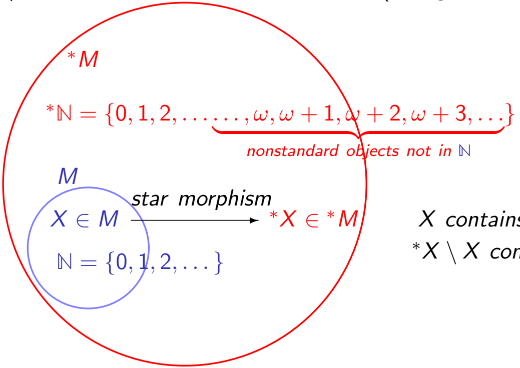

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3) Idealization/Saturation . . .

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And analogous results for fragments of IST.

A fragment based on Gödel's T

van den Berg, Briseid, Safarik, A functional interpretation of nonstandard arithmetic, APAL2012

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Only a finite sequence of witnesses; φ is internal.

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Same for nonstandard version H of E-HA$^\omega$ and intuitionistic logic.

A new computational aspect of NSA

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All theorems of PURE Nonstandard Analysis can be mined using the term extraction result (of P and H).

The unreasonable effectiveness of NSA

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From a proof that f is nonstandard uniformly continuous in P, i.e.

$$(\forall x, y \in [0,1])(x \approx y \to f(x) \approx f(y)), \tag{1}$$

we can extract a term t^1 (from Gödel's T) such that E-PA$^\omega$ proves

$$(\forall k^0)(\forall x, y \in [0,1])(|x-y| < \tfrac{1}{t(k)} \to |f(x)-f(y)| < \tfrac{1}{k}), \tag{2}$$

The unreasonable effectiveness of NSA

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From a proof that f is nonstandard uniformly continuous in P, i.e.

$$(\forall x, y \in [0,1])(x \approx y \to f(x) \approx f(y)), \tag{1}$$

we can extract a term t^1 (from Gödel's T) such that E-PA$^\omega$ proves

$$(\forall k^0)(\forall x, y \in [0,1])(|x - y| < \tfrac{1}{t(k)} \to |f(x) - f(y)| < \tfrac{1}{k}), \tag{2}$$

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Et pour les constructivists: la même chose!

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But (3) is the theorem expressing continuity implies Riemann integration from constructive analysis and computable math.

Explicit Reverse Mathematics

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The unreasonable effectiveness of NSA

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$=$ the EXPLICIT version of FAN $\leftrightarrow$ (cont $\to$ Rieman int. on $[0,1]$).

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...which is a theorem from constructive analysis and comp. math.

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Thus, NSA provides a 'computational foundation' (for sosoa).

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(5) is a thm from numerical analysis, called HERBRANDISATION of (4)

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we can extract terms i, o such that for all $f, g:\mathbb{R}\to\mathbb{R}$, and $\varepsilon'>0$:

$$(\forall x,y\in[0,1],\varepsilon>i(g,\varepsilon'))(|x-y|<g(\varepsilon)\to|f(x)-f(y)|<\varepsilon) \\ \downarrow \qquad (5) \\ (\forall \pi,\pi'\in P([0,1]))\big(\|\pi\|,\|\pi'\|<o(g,\varepsilon')\to|S_\pi(f)-S_{\pi'}(f)|\leq\varepsilon'\big)$$

is provable in E-PA$^\omega$, AND VICE VERSA: if E-PA$^\omega \vdash$ (5), then P $\vdash$ (4)

(5) is a thm from numerical analysis, called HERBRANDISATION of (4)

Every theorem of pure NSA has such a ‘meta-equivalent’ Hebrandisation.

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> Actually, the development of this particular example should not stop here, because whenever you have a theorem: B → A, then you suspect that you have a theorem: B is approximately true → A is approximately true. So there should be an even further development of this theory, namely to say what we mean for B to be approximately true, and then we have conjectured an implication, which we should try to prove. I have not done this, but it occured to me while preparing this talk that the conjecture is clear enough. I shall not take the time to present it here.

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on the same page of *Historia Mathematica* he trashes NSA.

A more recent attempt at mathematics by formal finesse is non-standard analysis. I gather that it has met with some degree of success, whether at the expense of giving significantly less meaningful proofs I do not know. My interest in non-standard

analysis is that attempts are being made to introduce it into calculus courses. It is difficult to believe that debasement of meaning could be carried so far.

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‘directly’ means that the meta-equivalence between a nonstandard thm and its Herbrandisation is acceptable to the finitist/constructivist.

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Note that the numerical version is satisfied by infinitely many terms i, o.

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Warning: Term extraction using M often produces vacuous truths (always for theorems requiring arithmetical comprehesion).

Impredicative, predicative and . . . locally constructive

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RM: (S^2) is equivalent to 'all sets are located'. We can replace locatedness by (S^2), while still obtaining computational info!

Final Thoughts

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Any questions?