Nonstandard Analysis as a computational foundation

Sam Sanders

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Computational Foundation?

Univalent foundations of mathematics is Vladimir Voevodsky’s new program for a comprehensive, *computational* foundation for mathematics based on the homotopical interpretation of type theory (aka HOTT).
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Subliminal message: ZFC, the ‘old’ foundation of mathematics is not ‘computational’, and therefore HOTT is better.
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Subliminal message: ZFC, the ‘old’ foundation of mathematics is not ‘computational’, and therefore HOTT is better.

In this talk, we show that Nonstandard Analysis provides ZFC with a ‘computational’ foundation.
Computational Foundation?

What is a ‘computational’ foundation?
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Computational Foundation?

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**Computational foundation:** HOTT is based on Martin-Löf’s intuitionistic type theory:
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**Computational foundation:** **HOTT** is based on **Martin-Löf’s intuitionistic type theory:** **BHK-interpretation** of constructive mathematics.
Computational Foundation?

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**NOT:** a computer implementation of mathematics: Wiedijk claims that **Mizar** has the largest library; Mizar is based on classical logic and an extension of ZFC.

**Computational foundation:** **HOTT** is based on Martin-Löf’s intuitionistic type theory: BHK-interpretation of constructive mathematics.

We show that Nonstandard Analysis provides a *similarly constructive* interpretation of mathematics. (Bishop and Connes)
A little test...\[\[
\text{Which statement has } \textbf{the most constructive/numerical content?}\]
A little test...

Which statement has the most constructive/numerical content?

\[(\forall x, y \in [0, 1]) [x \approx y \rightarrow f(x) \approx f(y)] \downarrow \]

\[(\forall \pi, \pi' \in P([0, 1])) (\|\pi\|, \|\pi'\| \approx 0 \rightarrow S_\pi(f) \approx S_{\pi'}(f))]\]
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Which statement has the most constructive/numerical content?

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\]

\[
\downarrow
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(\forall \pi, \pi' \in P([0, 1]))(\|\pi\|, \|\pi'\| \approx 0 \rightarrow S_\pi(f) \approx S_{\pi'}(f))
\]

OR

\[
(\forall k^0)(\forall x, y \in [0, 1])(|x - y| < \frac{1}{g(k)} \rightarrow |f(x) - f(y)| < \frac{1}{k})
\]

\[
\downarrow
\]

\[
(\forall k')(\forall \pi, \pi' \in P([0, 1]))(\|\pi\|, \|\pi'\| < \frac{1}{s(g, k')}) \rightarrow |S_\pi(f) - S_{\pi'}(f)| \leq \frac{1}{k'}
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A little test... 

Which statement has the most constructive/numerical content?

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\[(\forall k')(\forall \pi, \pi' \in P([0, 1]))(\|\pi\|, \|\pi'\| \leq \frac{1}{s(g,k')} \rightarrow |S_{\pi}(f) - S_{\pi'}(f)| \leq \frac{1}{k'})\],

As we will see: the first one! (up to finitistic manipulation)
Means to an end

Technical aim: To show that proofs of theorems of PURE Nonstandard Analysis can be mined to produce effective theorems not involving NSA, and vice versa.
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**Technical aim:** To show that proofs of theorems of PURE Nonstandard Analysis can be mined to produce *effective* theorems not involving NSA, and *vice versa*.

**PURE Nonstandard Analysis** = only involving the nonstandard definitions (of continuity, compactness, diff., Riemann int., ...)

**Effective theorem** = Theorem from constructive/computable analysis OR an (explicit) equivalence from Reverse Math.
Means to an end

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Vice versa? Certain effective theorems, called Herbrandisations, imply the nonstandard theorem from which they were obtained!
Means to an end

**Technical aim:** To show that proofs of theorems of **PURE Nonstandard Analysis** can be mined to produce **effective theorems** not involving **NSA**, and **vice versa**.

**PURE Nonstandard Analysis** = only involving the **nonstandard** definitions (of continuity, compactness, diff., Riemann int., ...)

**Effective theorem** = Theorem from constructive/computable analysis OR an (explicit) equivalence from Reverse Math.

**Vice versa?** Certain effective theorems, called **Herbrandisations**, imply the nonstandard theorem from which they were obtained!

**Motivation:** Many authors have observed the ‘constructive nature’ of the practice of NSA. (Horst Osswald’s **local constructivity**).
Introducing Nonstandard Analysis

Robinson’s semantic approach (1965):
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\[\mathbb{N} = \{0, 1, 2, \ldots\}\]

\[\ast \mathbb{N} = \{0, 1, 2, \ldots, \omega, \omega + 1, \ldots\}\]
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$\mathcal{M} = \{0,1,2,\ldots, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots\}$

$\mathbb{N} = \{0,1,2,\ldots\}$

$\mathcal{N}$ contains the nonstandard objects not in $\mathbb{N}$
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Robinson’s semantic approach (1965): For a given structure $M$, build $^\ast M \supseteq M$, a nonstandard model of $M$ (using free ultrafilter).

$^\ast \mathbb{N} = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots\}$

$^\ast X \in ^\ast M$

$X$ contains the standard objects

$X$ contains the nonstandard objects not in $\mathbb{N}$

$X$ contains the standard objects

star morphism

$\forall X \in M$
Robinson’s *semantic* approach (1965): For a given structure $M$, build $^*M \supseteq M$, a nonstandard model of $M$ (using free ultrafilter).

- $^\ast \mathbb{N} = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots\}$
- $^*X \in ^*M$ contains the *standard* objects
- $X \subseteq M$ contains the *nonstandard* objects
Robinson’s **semantic** approach (1965): For a given structure $M$, build $\ast M \supset M$, a nonstandard model of $M$ (using free ultrafilter).

Three important properties connecting $M$ and $\ast M$:
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Robinson’s semantic approach (1965): For a given structure $M$, build $^{*}M$ ⊇ $M$, a nonstandard model of $M$ (using free ultrafilter).

$^{*}M$ ⊇ $M$

$^{*}\mathbb{N} = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots\}$

nonstandard objects not in $\mathbb{N}$

$^{*}X \in ^{*}M$

$X$ contains the standard objects

$^{*}X \setminus X$ contains the nonstandard objects

Three important properties connecting $M$ and $^{*}M$:

1) Transfer $M \models \varphi \iff ^{*}M \models ^{*}\varphi$ $(\varphi \in L_{ZFC})$
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Robinson’s semantic approach (1965): For a given structure $M$, build $\mathcal{N} = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots\}$, a nonstandard model of $M$ (using free ultrafilter).

$\mathcal{N}$ contains the standard objects
$\mathcal{N} \cup \{\text{nonstandard objects not in } \mathcal{N}\}$

Three important properties connecting $M$ and $\mathcal{N}$:
1) Transfer $M \models \varphi \iff \mathcal{N} \models \ast \varphi$ ($\varphi \in L_{ZFC}$)
2) Standard Part $(\forall x \in \mathcal{N})(\exists y \in M)(\forall z \in M)(z \in x \iff z \in y)$
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Three important properties connecting $M$ and $^*M$:
1) Transfer $M \models \varphi \iff ^*M \models ^*\varphi$ ($\varphi \in L_{ZFC}$)
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Robinson’s **semantic** approach (1965): For a given structure \( M \), build \( *M \supseteq M \), a nonstandard model of \( M \) (using free ultrafilter).

\[ *\mathbb{N} = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots\} \]

Nonstandard objects not in \( \mathbb{N} \)

\[ X \in M \quad \text{star morphism} \quad *X \in *M \]

\( X \) contains the **standard** objects

\( *X \setminus X \) contains the **nonstandard** objects

Three important properties connecting \( M \) and \( *M \):
1) Transfer \( M \models \varphi \iff *M \models *\varphi \quad (\varphi \in L_{ZFC}) \)
2) Standard Part \((\forall x \in *M)(\exists y \in M)(\forall z \in M)(z \in x \iff z \in y)\) (reverse of *)
3) Idealization/Saturation . . .
Introducing Nonstandard Analysis

Nelson’s Internal Set Theory is a syntactic approach to Nonstandard Analysis.
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Add a new predicate $\text{st}(x)$ read ‘$x$ is standard’ to $L_{\text{ZFC}}$. 
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Add a new predicate $\text{st}(x)$ read ‘$x$ is standard’ to $L_{ZFC}$. We write $(\exists^{st} x)$ and $(\forall^{st} y)$ for $(\exists x)(\text{st}(x) \land \ldots)$ and $(\forall y)(\text{st}(y) \rightarrow \ldots)$. 
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Internal Set Theory IST is ZFC plus the new axioms:
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Internal Set Theory IST is ZFC plus the new axioms:

Transfer: $(\forall^{st} x) \varphi(x, t) \rightarrow (\forall x) \varphi(x, t)$ for internal $\varphi$ and standard $t$. 
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Internal Set Theory **IST** is ZFC plus the new axioms:

**Transfer:** $(\forall \text{st} x)\varphi(x, t) \rightarrow (\forall x)\varphi(x, t)$ for internal $\varphi$ and standard $t$.

**Standard Part:** $(\forall x)(\exists \text{st} y)(\forall \text{st} z)(z \in x \leftrightarrow z \in y)$. 
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Add a new predicate \( st(x) \) read ‘\( x \) is standard’ to \( L_{ZFC} \). We write \( (\exists^{st} x) \) and \( (\forall^{st} y) \) for \( (\exists x)(st(x) \land \ldots) \) and \( (\forall y)(st(y) \rightarrow \ldots) \). A formula is internal if it does not contain ‘\( st \)’; external otherwise.

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**Idealization:** \( \ldots \) (push quantifiers \( (\forall^{st} x) \) and \( (\exists^{st} y) \) to the front).
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Idealization:… (push quantifiers $(\forall^{st} x)$ and $(\exists^{st} y)$ to the front)

Conservation: ZFC and IST prove the same internal sentences.
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Internal Set Theory **IST** is ZFC plus the new axioms:

**Transfer:** $(\forall^{st} x)\varphi(x, t) \rightarrow (\forall x)\varphi(x, t)$ for internal $\varphi$ and standard $t$.

**Standard Part:** $(\forall x)(\exists^{st} y)(\forall^{st} z)(z \in x \leftrightarrow z \in y)$.

**Idealization:**… (push quantifiers $(\forall^{st} x)$ and $(\exists^{st} y)$ to the front).

**Conservation:** ZFC and IST prove the same internal sentences. And analogous results for fragments of IST.
A fragment based on Gödel’s T

van den Berg, Briseid, Safarik, A functional interpretation of nonstandard arithmetic, APAL2012
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$$(\forall^{\text{st}}x^\rho)(\exists^{\text{st}}y^\tau)\varphi(x, y) \to (\exists^{\text{st}}f^{\rho\to\tau^*})(\forall^{\text{st}}x^\rho)(\exists y^\tau \in f(x))\varphi(x, y)$$

Only a finite sequence of witnesses; $\varphi$ is internal.
A fragment based on Gödel’s $T$

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\[(\forall^{st} x^\rho)(\exists^{st} y^\tau) \varphi(x, y) \rightarrow (\exists^{st} f^\rho \rightarrow \tau^\ast)(\forall^{st} x^\rho)(\exists y^\tau \in f(x)) \varphi(x, y)\]

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No Transfer
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No Transfer

$P := E-PA^\omega + I + HAC_{int}$ is a conservative extension of $E-PA^\omega$. 
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Only a finite sequence of witnesses; $\varphi$ is internal.

No Transfer

$P := E-PA^\omega + I + HAC_{\text{int}}$ is a conservative extension of $E-PA^\omega$.

Same for nonstandard version $H$ of $E-HA^\omega$ and intuitionistic logic.
A new computational aspect of NSA
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TERM EXTRACTION
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If system P (resp. H) proves \((\forall^{st} x)(\exists^{st} y) \varphi(x, y)\) (\(\varphi\) internal)

OBSERVATION: Nonstandard definitions (of continuity, compactness, Riemann int., etc) can be brought into the 'normal form' \((\forall^{st} x)(\exists^{st} y) \varphi(x, y)\).

Such normal forms are closed under mode ponens (in both P and H)

All theorems of PURE Nonstandard Analysis can be mined using the term extraction result (of P and H).
A new computational aspect of NSA

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van den Berg, Briseid, Safarik, A functional interpretation of nonstandard arithmetic, APAL2012

If system P (resp. H) proves \((\forall^{st} x)(\exists^{st} y)\varphi(x, y)\) (\(\varphi\) internal)

then a term \(t\) can be extracted from this proof such that \(E-PA^\omega\) (resp. \(E-HA^\omega\)) proves \((\forall x)(\exists y \in t(x))\varphi(x, y)\).

(Compare to Gödel-Gentzen and H. Friedman translation for \(\Pi^0_2\)-formulas)

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then a term \(t\) can be extracted from this proof such that \(E-PA^\omega\) (resp. \(E-HA^\omega\)) proves \((\forall x)(\exists y \in t(x)) \varphi(x, y)\).

(Compare to Gödel-Gentzen and H. Friedman translation for \(\Pi_2^0\)-formulas)

OBSERVATION: Nonstandard definitions (of continuity, compactness, Riemann int., etc) can be brought into the ‘normal form’ \((\forall^{st} x)(\exists^{st} y) \varphi(x, y)\). Such normal forms are closed under modes ponens (in both P and H)

All theorems of PURE Nonstandard Analysis can be mined using the term extraction result (of P and H).
The unreasonable effectiveness of NSA

Example I: Continuity.
The unreasonable effectiveness of NSA

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From a proof that $f$ is nonstandard uniformly continuous in $P$, i.e.

$$(\forall x, y \in [0, 1])(x \approx y \rightarrow f(x) \approx f(y)),$$
The unreasonable effectiveness of NSA

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(\forall x, y \in [0, 1])(x \approx y \rightarrow f(x) \approx f(y)),
\]

we can extract a term \( t^1 \) (from Gödel’s T) such that \( \text{E-PA}^{\omega} \) proves

\[
(\forall k^0)(\forall x, y \in [0, 1])(|x - y| < \frac{1}{t(k)} \rightarrow |f(x) - f(y)| < \frac{1}{k}),
\]
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Example I: Continuity.

From a proof that $f$ is nonstandard uniformly continuous in $P$, i.e.

$$(\forall x, y \in [0, 1])(x \approx y \rightarrow f(x) \approx f(y)),$$  \hspace{1cm} (1)

we can extract a term $t^1$ (from Gödel’s T) such that $E\text{-PA}^\omega$ proves

$$(\forall k^0)(\forall x, y \in [0, 1])(|x - y| < \frac{1}{t(k)} \rightarrow |f(x) - f(y)| < \frac{1}{k}),$$  \hspace{1cm} (2)

AND VICE VERSA: $E\text{-PA}^\omega \vdash (2)$ implies $P \vdash (1)$. 

The unreasonable effectiveness of NSA

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(\forall x, y \in [0, 1])(x \approx y \rightarrow f(x) \approx f(y)), \quad (1)
\]

we can extract a term \( t^1 \) (from Gödel’s T) such that \( \text{E-PA}^\omega \) proves

\[
(\forall k^0)(\forall x, y \in [0, 1])(|x - y| < \frac{1}{t(k)} \rightarrow |f(x) - f(y)| < \frac{1}{k}), \quad (2)
\]

AND VICE VERA: \( \text{E-PA}^\omega \vdash (2) \) implies \( P \vdash (1) \).

(2) is the notion of continuity (with a modulus \( t \)) used in constructive analysis and computable math (Bishop, etc).
The unreasonable effectiveness of NSA

Example I: Continuity.

From a proof that $f$ is nonstandard uniformly continuous in $P$, i.e.

$$(\forall x, y \in [0,1])(x \approx y \rightarrow f(x) \approx f(y)),$$  \hspace{1cm} (1)

we can extract a term $t^1$ (from Gödel’s T) such that $E\text{-}PA^\omega$ proves

$$(\forall k^0)(\forall x, y \in [0,1])(|x - y| < \frac{1}{t(k)} \rightarrow |f(x) - f(y)| < \frac{1}{k}),$$ \hspace{1cm} (2)

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(2) is the notion of continuity (with a modulus $t$) used in constructive analysis and computable math (Bishop, etc).

Et pour les constructivists: la même chose!
The unreasonable effectiveness of NSA

Example II: Continuity implies Riemann integration
The unreasonable effectiveness of NSA

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From a proof that nonstandard uniformly continuity implies nonstandard Riemann integration in P, i.e.
The unreasonable effectiveness of NSA

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From a proof that nonstandard uniformly continuity implies nonstandard Riemann integration in P, i.e.

\[(\forall f : \mathbb{R} \to \mathbb{R})[(\forall x, y \in [0, 1])[x \approx y \to f(x) \approx f(y)] \downarrow \]

\[(\forall \pi, \pi' \in P([0, 1]))(||\pi||, ||\pi'|| \approx 0 \to S_\pi(f) \approx S_{\pi'}(f))\],

we can extract a term $s_2$ such that for $f : \mathbb{R} \to \mathbb{R}$ and modulus $g_1$:

\[(\forall k_0)(\forall x, y \in [0, 1])(|x - y| < 1 \to |f(x) - f(y)| < 1)\]

\[(\forall k') (\forall \pi, \pi' \in P([0, 1]))(||\pi||, ||\pi'|| < 1 \to |S_\pi(f) - S_{\pi'}(f)| \leq 1)\] (3)

is provable in $E-\text{PA}_\omega$.

(and the same for $E-\text{HA}_\omega$)
The unreasonable effectiveness of NSA

Example II: Continuity implies Riemann integration

From a proof that nonstandard uniformly continuity implies nonstandard Riemann integration in $P$, i.e.

$$(\forall f : \mathbb{R} \to \mathbb{R})(\forall x, y \in [0, 1])[x \approx y \rightarrow f(x) \approx f(y)]$$

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$$(\forall k^0)(\forall x, y \in [0, 1])(|x - y| < \frac{1}{g(k)} \rightarrow |f(x) - f(y)| < \frac{1}{k}) \tag{3}$$

$$\downarrow$$

$$(\forall k')(\forall \pi, \pi' \in P([0, 1]))(\|\pi\|, \|\pi'\| < \frac{1}{s(g, k')} \rightarrow |S_{\pi}(f) - S_{\pi'}(f)| \leq \frac{1}{k'})$$
The unreasonable effectiveness of NSA

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\[(\forall f : \mathbb{R} \to \mathbb{R}) \left[ (\forall x, y \in [0, 1]) [x \approx y \to f(x) \approx f(y)] \right] \]

\[
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\[(\forall \pi, \pi' \in P([0, 1])) (\|\pi\|, \|\pi'\| \approx 0 \to S_{\pi}(f) \approx S_{\pi'}(f)) \],

we can extract a term \( s^2 \) such that for \( f : \mathbb{R} \to \mathbb{R} \) and modulus \( g^1: \)

\[(\forall k^0) (\forall x, y \in [0, 1]) (|x - y| < \frac{1}{g(k)} \to |f(x) - f(y)| < \frac{1}{k}) \quad (3) \]

\[
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The unreasonable effectiveness of NSA

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is provable in E-PA\(^\omega\). (and the same for E-HA\(^\omega\))

But (3) is the theorem expressing continuity implies Riemann integration from constructive analysis and computable math.
Explicit Reverse Mathematics

Example III: The monotone convergence theorem
Explicit Reverse Mathematics

Example III: The monotone convergence theorem

From a proof in P of the following equivalence:

\[(\forall^{\text{st}} f^1)[(\exists n) f(n) = 0 \rightarrow (\exists^{\text{st}} m) f(m) = 0]\]

(\(\Pi^0_1\)-TRANS)

\[\iff\]

Every standard monotone sequence in \([0, 1]\) nonstandard converges
Explicit Reverse Mathematics

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The above is the EXPLICIT equivalence \(\text{ACA}_0 \iff \text{MCT}\).
Explicit Reverse Mathematics

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Explicit Reverse Mathematics

Example IV: Group Theory
Explicit Reverse Mathematics

Example IV: Group Theory

From a proof in P of the following equivalence:

\[(\forall^{st}f^1)[(\exists g^1)(\forall n)f(\overline{gn}) = 0 \rightarrow (\exists^{st} g^1)(\forall^{st} m)f(\overline{gm}) = 0] \tag{\Pi^1_1\text{-TRANS}}\]

\[\leftrightarrow\] Every standard countable abelian group is a direct sum of a standard divisible group and a standard reduced group
Explicit Reverse Mathematics

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Explicit Reverse Mathematics

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Explicit Reverse Mathematics

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The unreasonable effectiveness of NSA

Example V: Compactness
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$X$ is nonstandard compact IFF $(\forall x \in X)(\exists^{st} y \in X)(x \approx y)$.
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From a proof in P of the following equivalence:

$[0, 1]$ is nonstandard compact $(\text{STP})$

$\iff$

Every ns-cont. function is ns-Riemann integrable on $[0, 1]$
The unreasonable effectiveness of NSA

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If $\Omega^3$ is the fan functional, then $u(\Omega)$ computes the Riemann integral for any cont. function on $[0, 1]$. 

The unreasonable effectiveness of NSA

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$= \text{ the EXPLICIT version of } FAN \leftrightarrow (\text{cont } \to \text{ Rieman int. on } [0, 1])$. 
The unreasonable effectiveness of NSA

Example VI: Compactness bis
Compactness has multiple non-equivalent normal forms.
The unreasonable effectiveness of NSA

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Compactness has multiple non-equivalent normal forms. In Example V, the normal form of ns-compactness was a nonstandard version of FAN.
The unreasonable effectiveness of NSA

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Compactness has multiple non-equivalent normal forms. In Example V, the normal form of ns-compactness was a nonstandard version of FAN. Here, the normal form expresses ‘the space can be discretely divided into infinitesimal pieces’.
The unreasonable effectiveness of NSA

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Compactness has multiple non-equivalent normal forms. In Example V, the normal form of ns-compactness was a nonstandard version of FAN. Here, the normal form expresses ‘the space can be discretely divided into infinitesimal pieces’.

From a proof in P of the following theorem

For a uniformly ns-cont. $f$ and ns-compact $X$, $f(X)$ is also ns-compact.
The unreasonable effectiveness of NSA

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If $\Psi$ witnesses that $X$ is totally bounded and $g$ is a modulus of uniform cont. for $f$, then $u(\Psi, g)$ witnesses that $f(X)$ is totally bounded.
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If $\Psi$ witnesses that $X$ is totally bounded and $g$ is a modulus of uniform cont. for $f$, then $u(\Psi, g)$ witnesses that $f(X)$ is totally bounded.

...which is a theorem from constructive analysis and comp. math.
Conclusion

Nonstandard Analysis is unreasonably effective as follows:
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a) Focus on theorems of pure NSA, i.e. involving the nonstandard definitions of continuity, differentiation, Riemann integration, compactness, open sets, et cetera.
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b) **TERM EXTRACTION** works for **HUGE** class ‘theorems of pure NSA’
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a) Observation: Every theorem of pure NSA can be brought into the normal form \( (\forall^\text{st} x)(\exists^\text{st} y)\varphi(x, y) \) (\( \varphi \) internal).
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If P proves \((\forall^{st} x)(\exists^{st} y)\varphi(x, y)\), then from the latter proof, a term \(t\) can be extracted such that E-PA\(^{\omega}\) proves \((\forall x)(\exists y \in t(x))\varphi(x, y)\)
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Thus, NSA provides a ‘computational foundation’ (for sosoal).
Towards meta-equivalence: Hebrandisations

From a proof that nonstandard uniformly continuity implies nonstandard Riemann integration in $P$, i.e.

$$(\forall f: \mathbb{R} \to \mathbb{R}) \left( (\forall x, y \in [0, 1]) [x \approx y \to f(x) \approx f(y)] \right) \downarrow (4)$$

we can extract terms $i, o$ such that for all $f, g: \mathbb{R} \to \mathbb{R}$, and $\varepsilon' > 0$:

$$(\forall x, y \in [0, 1], \varepsilon > i(g, \varepsilon')) ((|x - y| < g(\varepsilon) \to |f(x) - f(y)| < \varepsilon)) \downarrow (5)$$

$$(\forall \pi, \pi' \in P([0, 1])) (\|\pi\|, \|\pi'\| < o(g, \varepsilon') \to |S_\pi(f) - S_{\pi'}(f)| \leq \varepsilon')$$

is provable in $E-\text{PA}_\omega$, AND VICE VERSA: if $E-\text{PA}_\omega \vdash (5)$, then $P \vdash (4)$

(5) is a thm from numerical analysis, called HERBRANDISATION of (4)

Every theorem of pure NSA has such a 'meta-equivalent' Hebrandisation.
Towards meta-equivalence: Hebrandisations

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\[
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\]

↓ (4)

\[
(∀π, π′ ∈ P([0, 1]))\left(∥π∥, ∥π′∥ ≈ 0 → S_{π}(f) \approx S_{π′}(f)\right),
\]

we can extract terms $i, o$ such that for all $f, g: \mathbb{R} → \mathbb{R}$, and $ε′ > 0$:

\[
(∀x, y ∈ [0, 1], ε > i(g, ε′))\left(∥x − y∥ < g(ε) → ∥f(x) − f(y)∥ < ε\right)
\]

↓ (5)

\[
(∀π, π′ ∈ P([0, 1]))\left(∥π∥, ∥π′∥ < o(g, ε′) → |S_{π}(f) − S_{π′}(f)| ≤ ε′\right)
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Towards meta-equivalence: Hebrandisations

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\[(\forall f : \mathbb{R} \rightarrow \mathbb{R})[(\forall x, y \in [0, 1])[x \approx y \rightarrow f(x) \approx f(y)]]\]

\[(\forall \pi, \pi' \in P([0, 1]))(\|\pi\|, \|\pi'\| \approx 0 \rightarrow S_{\pi}(f) \approx S_{\pi'}(f))\] \hspace{1cm} (4)

we can extract terms \(i, o\) such that for all \(f, g : \mathbb{R} \rightarrow \mathbb{R},\) and \(\varepsilon' > 0:\)

\[(\forall x, y \in [0, 1], \varepsilon > i(g, \varepsilon'))(|x - y| < g(\varepsilon) \rightarrow |f(x) - f(y)| < \varepsilon)\]

\[(\forall \pi, \pi' \in P([0, 1]))(\|\pi\|, \|\pi'\| < o(g, \varepsilon') \rightarrow |S_{\pi}(f) - S_{\pi'}(f)| \leq \varepsilon')\] \hspace{1cm} (5)

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Every theorem of pure NSA has such a ‘meta-equivalent’ Hebrandisation.
Application I: Cutting out the middle man in vagueness
Application I: Cutting out the middle man in vagueness

The predicate ‘\(\approx\)’ is the text-book formalisation of the vague notion ‘nearness’.
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Literally: ‘\(\approx\)’ from NSA has been used as a foundation for modelling vague predicates like nearness in AI, fuzzy set theory, and optimisation and control.
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The predicate ‘≈’ is the text-book formalisation of the vague notion ‘nearness’.

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Continuity in physics: If \(x, y\) are ‘very close’, so are their images.
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Actually, the development of this particular example should not stop here, because whenever you have a theorem: $B \rightarrow A$, then you suspect that you have a theorem: $B$ is approximately true $\rightarrow A$ is approximately true. So there should be an even further development of this theory, namely to say what we mean for $B$ to be approximately true, and then we have conjectured an implication, which we should try to prove. I have not done this, but it occurred to me while preparing this talk that the conjecture is clear enough. I shall not take the time to present it here.
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on the same page of *Historia Mathematica* he trashes NSA.

A more recent attempt at mathematics by formal finesse is non-standard analysis. I gather that it has met with some degree of success, whether at the expense of giving significantly less meaningful proofs I do not know. My interest in non-standard analysis is that attempts are being made to introduce it into calculus courses. It is difficult to believe that debasement of meaning could be carried so far.
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Herbrandisations lead to a rather structuralist view of mathematics:

The objects of mathematics do not matter, but mathematical structures do.
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In particular, Herbrandisations give a way of talking ‘directly’ about Nonstandard Analysis in the standard model.
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Herbrandisations lead to a rather **structuralist** view of mathematics:

The objects of mathematics do not matter, but mathematical structures do.

In particular, Herbrandisations give a way of talking ‘directly’ about Nonstandard Analysis in the standard model.

‘directly’ means that the meta-equivalence between a nonstandard thm and its Herbrandisation is acceptable to the finitist/constructivist.
Application III: Frege’s Sinn und bedeutung
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**Sinne** $\approx$ the way a term refers to an object
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Clark Kent and Superman refer to the same person (same Bedeutung). However, they do so in a very different way (different Sinne)
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The nonstandard theorem = the Bedeutung

The Hebrandisation/numerical version = the Sinne
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The nonstandard theorem $=$ the Bedeutung

The Hebrandisation/numerical version $=$ the Sinne

Note that the numerical version is satisfied by infinitely many terms $i, o$.  
Mining standard proofs

Question: Can you also mine proofs not involving NSA?
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Warning: Term extraction using M often produces vacuous truths (always for theorems requiring arithmetical comprehension).
Impredicative, predicative and ... locally constructive
Impredicative, predicative and . . . locally constructive

The Suslin functional \((S^2)\) is the functional version of \(\Pi^1_1\)-CA\(_0\):

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(\exists S^2)(\forall f^1)[S(f) = 0 \leftrightarrow (\exists g^1)(\forall n^0)(f(\overline{g}n) = 0)]. \tag{S^2}
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Standard objects in \(P\) and \(H\) are those which are computationally relevant (cf. Berger’s uniform HA and Lifschitz’s calculable numbers)

RM: \((S^2)\) is equivalent to ‘all sets are located’. We can replace locatedness by \((S^2)\), while still obtaining computational info!
Final Thoughts
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Any questions?