What are Axioms of Set Theory?

Set-theorists use the term *Axiom (of Set Theory)* quite freely.

What do they mean by it?

Examples

Axioms of ZFC:

Axiom of Extensionality
Pairing Axiom
Separation Axiom
Union Axiom
Powerset Axiom
Axiom of Infinity
Replacement Axiom
Axiom of Foundation
Axiom of Choice

What are Axioms of Set Theory?

Beyond ZFC:

Axiom of Constructibility
Axiom of Determinacy
Large Cardinal Axioms
Reinhardt's Axiom
Cardinal Characteristic Axioms
Martin's Axiom
Axiom A
Proper Forcing Axiom
Open Colouring Axiom

What are Axioms of Set Theory?

Hypotheses:

Continuum Hypothesis
Suslin Hypothesis
Kurepa Hypothesis
Singular Cardinal Hypothesis

Principles:

Diamond Principle
Square Principle
Vopenka's Principle
Reflection Principles

What are Axioms of Set Theory?

When does a statement achieve the status of *axiom*, *hypothesis* or *principle*?

It is worth examining this question in three specific cases:

A. The Pairing Axiom
B. Large Cardinal Axioms
C. The Axiom of Choice

What are Axioms of Set Theory?

A. The Pairing Axiom

"If x, y are sets then there is a set whose elements are precisely x and y"

Such an assertion is basic to the way sets are regarded in mathematics. Moreover if the term "set" is used in a way that violates this assertion we would have to regard this use as based upon a different concept altogether.

Thus, as Feferman has observed, the Pairing Axiom qualifies as an axiom in the ideal sense of the Oxford English Dictionary:

"A self-evident proposition requiring no formal demonstration to prove its truth, but received and assented to as soon as mentioned."

In this sense the Pairing Axiom is *intrinsic* to the concept of set.

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B. Large Cardinal Axioms

I refer here to axioms such as:

There is an inacessible cardinal
There is a measurable cardinal
There is a supercompact cardinal

Why do these gain the status of axioms?

The reason is that these assertions play an important role in the mathematical development of set theory:

What are Axioms of Set Theory?

Inaccessibles yield the consistency of every definable set of reals being Lebesgue measurability

Measurables yield a violation of the Axiom of Constructibility

Supercompacts allow one to violate the Singular Cardinal Hypothesis

Indeed large cardinal axioms are essential to our understanding of modern set theory

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C. The Axiom of Choice

There is disagreement as to whether AC is intrinsic to the concept of set.

But intrinsicness is not the only source for its appeal: the Axiom of Choice has myriad uses across many areas of mathematics, facilitating many arguments that would not otherwise be possible.

Thus in contrast to Large Cardinal Axioms, AC is of great value for developing a set-theoretic foundation of mathematics as a whole, and not just for set theory itself.

What are Axioms of Set Theory?

In summary, we have:

Large Cardinal Axioms: Useful for the mathematical development of Set Theory

Axiom of Choice: Useful for the foundation of mathematics as a whole, not just for Set Theory

Axiom of Pairing: Intrinsic to the concept of set

For brevity, let's call these three types *axioms of practice, foundational axioms and intrinsic axioms*, keeping in mind that we are just talking about axioms of Set Theory.

Do the other examples mentioned fit into these categeories?
I believe that they do. As a preliminary classification:

What are Axioms of Set Theory?

Axioms of Practice:

Axiom of Constructibility
Axiom of Determinacy
Large Cardinal Axioms
Reinhardt's Axiom
Cardinal Characteristic Axioms
Martin's Axiom
Axiom A
Proper Forcing Axiom
Open Colouring Axiom
Continuum Hypothesis
Suslin Hypothesis, Kurepa Hypothesis
Singular Cardinal Hypothesis
Diamond Principle, Square Principle
Vopenka's Principle, Reflection Principles

What are Axioms of Set Theory?

Foundational:

Axiom of Choice

Intrinsic:

Axiom of Extensionality
Pairing Axiom
Separation Axiom
Union Axiom
Powerset Axiom
Axiom of Infinity
Replacement Axiom
Axiom of Foundation

What are Axioms of Set Theory?

Now note the following about these lists:

There are many *axioms of practice* but only one *foundational axiom* and the *intrinsic axioms* all come from ZFC!

The list of axioms of practice can be extended much further, with set-generic absoluteness principles, P_{max}-axioms, an ultimate-L axiom, further forcing axioms like MM, etc.

But two obvious questions arise:

Question 1. Are there other *foundational axioms*, i.e. axioms other than AC which help provide a good foundation for mathematics outside of Set Theory?

Question 2. Are there other *intrinsic axioms*, inherent in the concept of set?

What are Axioms of Set Theory?

Recently, candidates for new *foundational axioms* are starting to emerge.
An example is PFA, the Proper Forcing Axiom (already listed as an *axiom of practice*).

PFA implies:

The Kaplansky Conjecture on Banach Algebras.
Every automorphism of the Calkin Algebra is inner.
All automorphisms of $P(\omega)$ *mod Fin* are trivial.
There is a non-free Whitehead group.
There are no Suslin lines.
and more ...

Does this qualify PFA as a *foundational axiom*?

Surely not yet.

What are Axioms of Set Theory?

AC was adopted due to its capability of facilitating a broad spectrum of results across mathematics, without contradicting any similarly attractive alternative axiom.

But PFA contradicts, for example, $V = L$, which also facilitates a broad spectrum of results across mathematics.

Question for the future: Which of PFA and $V = L$ is more "congenial" to mathematical practice, yielding results which are welcome to the mathematical community?

It is impossible to answer this question without a great deal of further exploration of the consequences of these axioms and an in-depth discussion of the motivations and perspectives of mathematicians. What is needed is:

What are Axioms of Set Theory?

A foundation based on mathematical practice. An exploration of the axioms of Set Theory that best serve the practice of mathematics outside of Set Theory.

It may also be that for some areas of mathematics, a foundation based on something other than Set Theory is most appropriate (consider HoTT).

In any case, we need a closer cooperation between logicians and mathematicians to sort this out.

What are Axioms of Set Theory?

Now we consider:

Question 2. Are there other *intrinsic axioms*, inherent in the concept of set?

The *Hyperuniverse Programme (HP)* takes the the *maximality* of the set-theoretic universe, as expressed through the *maximal iterative concept of set (MIC)*, to be inherent in the set-concept.

Maximality as expressed by the *MIC* comes in two forms:

Ordinal or height maximality: The universe V is "as tall as possible", i.e. the sequence of ordinals is "as long as possible".

Powerset or width maximality: The universe V is "as wide (or thick) as possible", i.e. the powerset of each set is "as large as possible".

What are Axioms of Set Theory?

The programme then embarks on an extensive analysis of the different mathematical formulations of height and width maximality, as well as "syntheses" of these formulations.

It is important to note three things. First:

Non-first-order: The *HP* formulations of maximality are almost never first-order, instead they are first-order in a mild "lengthening" of V, the least admissible set past V.

Indeed it appears that first-order formulations of *maximality* are either quite weak or poorly motivated.

What are Axioms of Set Theory?

Second, a key step in the programme is to argue that maximality criteria for V can in fact be analysed in terms of *countable* models of ZFC, the collection of which is called the *Hyperuniverse*.

This “reduction to the Hyperuniverse” makes use of V-logic and the downward Löwenheim-Skolem Theorem.

And third, the programme aims for the following conjecture:

An Optimal Maximality Criterion: Through the *HP* it will be possible to arrive at an optimal (*non-first-order*) criterion expressing the maximality of the set-theoretic universe in height and width; this criterion will have first-order consequences contradicting *CH*.

The *HP* is work in progress and it will take time to generate and analyse the full spectrum of maximality criteria.
Some of the criteria currently under consideration are:

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#-generation (height maximality)
IMH (Inner Model Hypothesis; a form of width maximality)
IMH# (a synthesis of #-generation and the IHH)
SIMH (Strong IMH)
SIMH# (a synthesis of SIMH and #-generation)
CardMax (related to the SIMH)
Width Reflection
Width Indiscernibility
Omniscience

and “syntheses” of the above.

When is an Axiom true?

Now we come to a controversial point.

Question. When is an axiom true?

I think that there has been enormous confusion about this, caused by a failure to distinguish different kinds of evidence for truth.

To clarify this point, let's return to the three examples mentioned earlier:

The Axiom of Pairing
The Axiom of Choice
Large Cardinal Axioms

When is an Axiom true?

The Axiom of Pairing is true.

Why do we believe this?

It is because it is an inherent feature of the concept of set as we understand it. Any two sets can be joined to form a new set; otherwise we have a different concept.

When is an Axiom true?

The Axiom of Choice is true.

This is said explicitly and often by set-theorists. Why?

Some would argue that it too is inherent in the set-concept. But I don't subscribe to this view, as in general we cannot explicitly describe a choice function on a collection of nonempty sets and therefore have no immediate argument that such a choice function should exist.

Rather, I think that AC is true because of its compelling importance for set theory *as a foundation for mathematics*. As said before, it has myriad uses across many areas of mathematics, facilitating many arguments that would not otherwise be possible. No compelling alternative to AC has arisen and AC has been definitively added to our list of standard axioms for Set Theory for this reason.

When is an Axiom true?

Large Cardinal Axioms are true.

How can one argue this?

It is clear that large cardinal axioms are essential to many working set-theorists, as they provide the consistency strengths needed to analyse many set-theoretic problems.

But consistency proofs typically start with large cardinals and then create a larger universe in which the given large cardinal properties are destroyed. From this we don't infer the existence of large cardinals in V but only in inner models of V.

Small large cardinal axioms are derivable from reflection, a form of height maximality. This is the case for example with inaccessible or weakly compact cardinals.

When is an Axiom true?

But the most natural form of width maximality, the IMH, contradicts large cardinal existence. So it is not at all clear how to derive large cardinal axioms from maximality. More generally, it is hard to argue that they are "inherent" in the set-concept.

Are Large Cardinal Axioms important for mathematics outside of Set Theory? There is very little evidence for this. Admittedly, they are useful in category theory and in non first-order model theory, but the sum total of their uses outside of Set Theory is quite small. Maybe this will change, but it doesn't look like it.

This leads to the conclusion: The evidence for the truth of Large Cardinal Axioms derives exclusively from their utility for the development of Set Theory as a branch of mathematics.

When is an Axiom true?

The above conclusion applies to most current claims about truth in Set Theory.

When Steel and Woodin talk about the "truth" of some version of the Ultimate-L axiom, they mean nothing more than "an attractive hypothesis to use when working in some areas of Set Theory". When others claim that Set-Generic Absoluteness Axioms are true they again just mean that they are useful in this sense.

A more robust view of set-theoretic truth is expressed by the following scenario from my Chiemsee paper:

When is an Axiom true?

Thesis of Set-Theoretic Truth. There will be first-order statements of set theory that well serve the needs of set-theoretic practice and of resolving independence across mathematics, and which are derivable from the maximality of the set-theoretic universe in height and width. Such statements will come to be regarded as *true statements of set theory*.

I hope that I'm right!