(Towards)
A Multiverse Induction Framework

Emil Weydert

Individual and Collective Reasoning Group
University of Luxembourg

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Outlook

- Looking for axioms
- Nonmonotonic inference
- Axiom induction
- Multiverse views
- Framework fragments
- Principles and problems
- Conclusions

Prologue

State of affairs: *Set theory is a bit special ...*

- Major foundational role
- Material incompleteness (annoying independence)
- Conceptual incompleteness (maximality failures)
- But: Platonist feelings very present
- But also: a well-structured, well-motivated space of robust axiomatic extensions

Resulting tasks:

- Further search for new axioms and criteria justifying them
- Understanding to what extent this process may/should converge (monism) resp. diverge (pluralism)

The sources

Ingredients and factors: for axiom discovery and justification

- Philosophical analysis (principles, paradigms, questions)
- Mathematical context (formal facts, recognized playgrounds)

  Zoo: V=L, Woodin, huge, IMH, $\text{IMH}^{\#}$, AD, PFA, ...
- Practical/technical desires or constraints

  (e.g. problem-solving potential, recursive axiomatizations)
- Aesthetical or cognitive reasons (depth, simplicity, pleasure ...)
- ... Platonic mathematical reality (invisible hand, Goedel-perception)

The process

Tasks: seeking, understanding, and evaluating candidates

- Candidates: principle instances, weaken/strengthen, combine ...
- Determine relationships: relative consistency, embeddability, ...
- Conceptual pros/cons: intuition, principles
- Technical pros/cons: usability, fertility, consequences
- Status decisions:
  - Special interest axioms?
  - Set-theoretic canon?

Idea: Axiom generation as a generalized inferential process

Formal axiom induction

Induction: Ampliative inference of defendable assumptions

Axiom induction in set theory:
A systematic logic-oriented approach for identifying and judging possible new axioms/truths for set theory, exploiting a formal model of set theoretic culture (principles, practice, knowledge)

Expectations:

- More clarity and unity given divergent motivations/views
- Formal procedures easier to compare, evaluate, and improve
- New higher-order criteria for axiom choice, i.e.

$\rightarrow$ **Inferential-level rationality postulates:**

- *Better axioms through better search methods*

What is classical inference?

What is a logic? $\mathcal{L} = (L, \vdash, \models)$

- Language L + Inference relation $\vdash \subseteq 2^L \times L$, resp.
  + Inference operator $C(\Sigma) = \{\psi \mid \Sigma \vdash \psi\}$
- Satisfaction relation $\models \subseteq Mod \times L$

Classical inference: standard specifications

- $\Sigma \vdash_{\mathcal{R}} \psi$ iff there is an $\mathcal{R}$-proof of ψ from Σ (deduction)
- $\Sigma \vdash_{sem} \psi$ iff $Mod_{\models}(\Sigma) \subseteq Mod_{\models}(\{\psi\})$ (semantic entailment)

Tarski's abstract principles: verified by $\vdash_{\mathcal{R}}, \vdash_{sem}$

1. *Inclusion*: $\Sigma \subseteq Cn(\Sigma)$
2. *Idempotence*: $Cn(Cn(\Sigma)) = Cn(\Sigma)$
3. *Monotony*: $\Sigma \subseteq \Sigma'$ implies $Cn(\Sigma) \subseteq Cn(\Sigma')$

What is generalized inference?

Knowledge representation in AI:
How to formally model commonsense reasoning?

Since the 80s substantial work on generalized notions of inference (a useful way to look at things) $\rightarrow$ violate Tarski's principles

- Induction (observations $\rightarrow$ theories): *Incl, Idemp?*
- Abduction (observations, theories $\rightarrow$ explanantions): *Incl*
- Closed world reasoning, negation by failure: *Incl*
- Plausible/default reasoning: *Incl, Idemp?*
- Inconsistency repair/aggregation: *Idemp*

All nonmonotonic: $\Sigma \mathrel{|\!\sim} \psi$ but $\Sigma \cup \Sigma' \mathrel{|\!\not\sim} \psi$

Instances of nonmonotonic logics

Nonmonotonic logic: $(L, L^*, \vdash, \models, \mid\!\sim)$

- L base language - L^* control language (e.g. prefs, defaults)
- $\vdash$ monotonic inference: $\Sigma \vdash \psi$ for $\Sigma \subseteq L$
- $\mid\!\sim$ nonmonotonic inference: $\Sigma \cup \Delta \mid\!\sim \psi$ for $\Delta \subseteq L^*$

$$\Sigma \cup \Delta \mid\!\sim \psi \text{ resp. } \Sigma \mid\!\sim_\Delta \psi$$

Default reasoning: defeasible conditionals $\varphi \rightsquigarrow \psi \in L^*$:

- *Plausible:* $\varphi \wedge \neg\psi$ less plausible than $\varphi \wedge \psi$
- *Context-based:* if φ, and ψ is possible/consistent, then ψ

Specificity pattern: validity depends on $\rightsquigarrow$ reading

- $\{\varphi\} \cup \{\varphi \rightarrow_{strict} \varphi', \varphi' \rightsquigarrow \psi, \varphi \rightsquigarrow \neg\psi\} \mid\!\sim \neg\psi$

Principles for nonmonotonic reasoning I

[Gabbay 85, Kraus, Lehmann, Magidor 90, Makinson 94]:

1. Basic nonmonotonic inference:

Reflexivity: $\psi \in \Sigma \ \Rightarrow \ \Sigma \mid\!\sim_\Delta \psi$

Left Equivalence: $\Sigma \dashv\vdash \Sigma', \Sigma \mid\!\sim_\Delta \psi \ \Rightarrow \ \Sigma' \mid\!\sim_\Delta \psi$

Right Weakening: $\Sigma \mid\!\sim_\Delta \psi, \ \psi \vdash \psi' \ \Rightarrow \ \Sigma \mid\!\sim_\Delta \psi'$

Right Conjunction: $\Sigma \mid\!\sim_\Delta \psi, \Sigma \mid\!\sim_\Delta \psi' \ \Rightarrow \ \Sigma \mid\!\sim_\Delta \psi \wedge \psi'$

Principles for nonmonotonic reasoning II

2. Cumulative nonmonotonic inference: $\ldots +$

Cautious Monotony: $\Sigma \mathrel{|\!\sim}_\Delta \varphi, \Sigma \mathrel{|\!\sim}_\Delta \psi \;\Rightarrow\; \Sigma \cup \{\varphi\} \mathrel{|\!\sim}_\Delta \psi$

Cautious Transitivity: $\Sigma \mathrel{|\!\sim}_\Delta \varphi, \Sigma \cup \{\varphi\} \mathrel{|\!\sim}_\Delta \psi \;\Rightarrow\; \Sigma \mathrel{|\!\sim}_\Delta \psi$

3. Preferential nonmonotonic inference: $\ldots +$

Reasoning by cases:

$$\Sigma \cup \{\varphi\} \mathrel{|\!\sim}_\Delta \psi, \Sigma \cup \{\varphi'\} \mathrel{|\!\sim}_\Delta \psi \;\Rightarrow\; \Sigma \cup \{\varphi \vee \varphi'\} \mathrel{|\!\sim}_\Delta \psi$$

Note: This is just a small sample. Also these principles ignore Δ, where the fun starts

Preferred model theory

How to specify nonmonotonic inference notions?

Simplest idea: use preference relation $\preceq$ on (partial) models

- $\Phi \mathrel{|\!\sim} \psi$ iff ψ-models "$\preceq$-minimal" within Φ-models

$\rightarrow$ basic inference is supraclassical: $\Sigma \vdash \psi$ implies $\Sigma \mathrel{|\!\sim}_\Delta \psi$

$\rightarrow$ $\mathrel{|\!\sim}$ is basic for binary relations, cumulative for preorders over partial models, and preferential for preorders over total models.

Beyond this: second-order preferences ($\preceq$ is here implicit)

- $\varphi \mathrel{|\!\sim}^Z_\Delta \psi$ iff $\varphi \leadsto \psi$ holds in all the preference-maximizing total preorder models of Δ

Axiom induction: Input/Output

Input: *A formal proxy of current set-theoretic context or culture: axioms, knowledge, practice, principles*

Community attitudes: epistemic, deontic, comparative, ...

- *Known*: accepted axioms $\Sigma = ZFC \cup X$ (ideally recursive) (more fine-grained: known theorems, proofs, ...)
- *Beliefs*: consistency assumptions, compatibilities Θ
- *Possible*: axiom sets of interest $\Gamma_i \subset L(\in)$ $(i \in I)$
- *Desirable*: instances of maximality principles Γ_i $(i \in I^+)$
- *Relations*: e.g. intrinsic/extrinsic preferences $(\preceq_j \mid j \in J)$

  (rel. cons., dominance, interpretative power, maximality, ...)

Within a multiverse: even more such relations!

Axiom induction:

- $\mathcal{I}$: $(\Sigma, Context) \rightarrow \Sigma^* \subseteq L(\in)$

Simple model: $Input = (\Sigma, \Gamma, \Theta, \preceq_j)$

Γ: relevance, Θ: incompatibility, $\preceq_j$: preorders

Note: $\Sigma \cup \Gamma$ not necessarily consistent

Reality more general: $\mathcal{I}'(Input) = (\Sigma^*, \Gamma^*, \Theta^*, \preceq_j^*)$

But: $\mathcal{I}'$ monotonic in Σ - not given for $\Gamma, \Theta, \preceq_j$

Multiverse

Myriads of independence results, difficulties to settle fundamental set-theoretic questions, philosophical modesty, anti-realism:

Pluralism: No unique universe of sets but a multiverse, a plurality of - at least - a priori equally legitimate universes

Multiverse: different forms

- philosophical/instrumental motivations
- realist/not
- general/specific (e.g. set-generic)
- top-down/bottom-up

Hyperuniverse Program

Hyperuniverse $\mathbb{H}$: All the countable transitive models of ZFC

The Hyperuniverse Program: pragmatic pluralism

Specify natural preferences over the hyperuniverse and use the first-order theory shared by the preferred universes to identify new, possibly intrinsically motivated set-theoretic truths/axioms

In particular, maximality principles (vertical/horizontal, combined) are translated into formal constraints over the hyperuniverse

Resulting axiom candidates can then be judged by conceptual or practical desiderata of set-theoretic practice

Utopia? An optimal maximality principle $\rightarrow$ canonical axiom set

Beyond the Hyperuniverse

Guiding idea: When trying to specify preferred universes, take seriously the full higer-order structure over universes:

$\rightarrow$ background universe $(W, \in) \models \hat{\Sigma} \supseteq ZFC$, if not $\hat{\Sigma} \supseteq \Sigma$

$\rightarrow$ bigger multiverse $\mathbb{MV}$ (extending $\mathbb{H}$)

$\rightarrow$ may exploit a richer structure among universes

Classical Hyperuniverse:
$\mathbb{H}$ is a minimal instance in line with some multiverse priorities from the early days (forcing, critical w.r.t. platonism, one-shot)

Some desiderata

Minimal domain requirements: e.g.

- $\mathbb{MV}$ is a class Φ^W_{MV} in $(W, \in)$
- $\mathbb{MV} \supseteq$ countable transitive models of ZF in W
- $\mathbb{MV} \subseteq$ transitive models of ZF in W
- $\mathbb{MV}$ is closed under inner class/set models of ZF
- $M \subseteq_{fin} \mathbb{MV}$ definable (with param.) in some $m_M \in \mathbb{MV}$

Exploited relations: From local absolute up to definable over $(W, \in)$ - for specifying construction, dominance, preference, modal accessibility relations

Why generalizing?
Useful to handle axioms in infinitary or higher-order logic

Towards MV-based axiom induction

Mathematical input context: $(\Sigma, \Gamma_i, \Theta_h, \preceq_j)$

$ZFC \subseteq \Sigma$ (current axioms): fix the initially accepted universes

$(\Gamma_i, \Theta_h, \preceq_j)$: some characteristics of the mathematical context

Formal multiverse context: $(\hat{\Sigma}, \Phi_{MV})$

$\hat{\Sigma}$ constrains $(W, \in)$, Φ_{MV} describes the multiverse domain

In the multiverse framework, we may for instance state:

- $\mathbf{\Gamma} = \{\Gamma_i \mid i \in I\}$ collection of *relevant* axiom sets to be realized by transitive models in $\mathbb{MV}$
- $\mathbf{\Theta}$ indicates incompatible sets, without suitable models in $\mathbb{MV}$

Axiom induction as nonmontonic inference

Should we encode the mathematical context in an extended language L^*, itself subjected to nonmontonic reasoning?

More difficult - whereas the accepted axiom set - except if it is discovered to be inconsistent - will normally grow, this may fail for the context components

Initial focus: inference relations for axiom induction: $\mid\!\sim^{\in}_{\Xi}$

parametrized by the global context $\Xi = (\hat{\Sigma}, \Phi_{MV}, \Gamma_i, \Theta_h, \preceq_j)$

Critical issues:

Peculiarities and needs are illustrated by the handling of standard postulates for nomonotonic reasoning

$\rightarrow$ important rationality conditions, but, as in default reasoning, by themselves no guarantee for the adequacy of the results

Postulates and axiom induction

Reflexivity: $\psi \in \Sigma \;\Rightarrow\; \Sigma \mathrel{|\!\!\sim}_{\Xi} \psi$

Axiom induction stipulates that, as long as Σ is assumed consistent, the content of the accepted axioms cannot be questioned. However, one could imagine that in practice, the discovery of a new unifying axiom subsuming previous ones would suggest syntactic change - a move which in a dynamic context is however not necessarily innocent.

Left Equivalence: $\Sigma \dashv\vdash \Sigma', \Sigma \mathrel{|\!\!\sim}_{\Delta} \psi \;\Rightarrow\; \Sigma' \mathrel{|\!\!\sim}_{\Delta} \psi$

Because the form of the axioms may suggest the strengthening of a specific axiom, a reformulation could hide this possibility and thereby affect the evolution of axioms. That is, the pointers of the axioms for nonmonotonic inference may fail to be invariant under logically equivalent premise change.

Right Weakening: $\Sigma \mid\!\sim_\Delta \psi,\ \ \psi \vdash \psi' \ \Rightarrow\ \Sigma \mid\!\sim_\Delta \psi'$

This postulate should certainly hold if the goal is to find new true mathematical statements. It is less appropriate if the target is to actually identify new axioms, which should also be insightful, simple, and strong enough. This, in addition to the fact that these features are hard to encode, may be a reason to focus on generating true statements.

Right Conjunction: $\Sigma \mid\!\sim_\Delta \psi, \Sigma \mid\!\sim_\Delta \psi' \ \Rightarrow\ \Sigma \mid\!\sim_\Delta \psi \wedge \psi'$

There are at least two readings of axiom induction: recommending new assertions or filtering inappropriate ones - which is not the same as proposing their negation (see CH). The first situation corresponds to proper inference and supports RC. The second one may well produce inconsistent proposals. If this occurs, one could let $\mathcal{I}$ offer collections of axiom sets. The failure of RC would be an indicator of pluralism. The problem is that this may no be obvious and thereby promote the acceptance of inconsistencies.

Cautious Monotony: $\Sigma \mid\!\sim_\Delta \varphi, \Sigma \mid\!\sim_\Delta \psi \;\Rightarrow\; \Sigma \cup \{\varphi\} \mid\!\sim_\Delta \psi$

The failure of cautious monotony means that two new assertions may be added together, but if we add only one of them, the addition of the other one could be blocked - a rather odd behaviour. This induces order dependence which becomes virulent because the addition of new axioms often proceeds through analysing proposals produced by rather unpredictable mechanisms linked to human creativity.

Cautious Transitivity: $\Sigma \mid\!\sim_\Delta \varphi, \Sigma \cup \{\varphi\} \mid\!\sim_\Delta \psi \;\Rightarrow\; \Sigma \mid\!\sim_\Delta \psi$

The failure of cautious transitivity, or reasoning by lemmata, reflects a weakening of inferential links by chaining. For instance, in a naive approach, an axiom may be accepted if it stays consistent with most of the relevant axioms in Γ. By iterating this step, eventually the forthcoming new axioms may come to share only a small part of relevant statements with the initial one.

Contributions, lessons, and perspectives

- A survey of some choices linked to formalizing axiom induction
- Elements of a formal framework for describing, developing, investigating and understanding formal accounts of axiom search in set theory
- A bridge to other systematic approaches to plausible reasoning in formal contexts
- A collection of critical issues to be addressed when specifying different types of axiom induction
- Suitable inferential principles for axiom induction can be considered when choosing among axiom search methods - they complement existing approaches even if their actual relevance is still unclear and needs to be investigated