Category forcings and generic absoluteness: Explaining the success of strong forcing axioms

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I will outline two points which explain why forcing axioms are so successful in solving problems and can be viable candidates to strenghten ZFC.

> *Forcing axioms spring from the same lines of thought which brought mathematicians to accept nonconstructive principles such as the axiom of choice* AC *or Baire's category theorem* BCT *and can be seen as natural generalizations of both of the latter statements.*

To some extent forcing axioms also generalize the notion of large cardinal axiom, but I will leave this latter remark a bit on the side of this talk.

The use of the axiom of choice and of Baire's category theorem is ubiquitous in mathematics. Some sparse examples:

- In algebra without the axiom of choice we cannot prove the existence of maximal ideals for many rings R.
- In functional analysis without the axiom of choice we cannot prove the Hahn-Banach extension theorem for continuous linear functionals defined on a closed subspace of an arbitrary Banach space.
- In functional analysis without Baire's category theorem we cannot prove the closed-graph theorem which characterize the continuity of a linear functional between Banach spaces by the request that its graph is a closed subspace of the product of its domain and codomain.
- ...

In principle most of mathematics can be developed without the full axiom of choice (the axiom of countable dependent choices in most cases suffices). However resorting to the axiom of choice simplify considerably the arguments and gives a neater picture of what's going on.

The use of AC and BCT is in all cases aimed at proving (non constructively) the existence of witnesses for a given mathematical property.

> *Forcing axioms provide powerful tools to produce existential witnesses of a given mathematical property.*

The second argument I want to bring forward regarding the usefulness of forcing axioms is that these axioms provide new proof tools to solve certain type of mathematical problems:

> *Generic absoluteness results for the theory of strong forcing axioms explain why forcing axioms are so effective and succesful in solving problems of third order arithmetic.*

Exactly in the same way Woodin's generic absoluteness result for $L(\mathbb{R})$ explain why large cardinals and determinacy axioms are so effective in settling the theory of projective sets of reals (i.e. second order arithmetic).

Outline

The axiom of choice is a global forcing axiom

This argument has been handled to me by Stevo Todorčević.

Definition

Let λ be an infinite cardinal. DC_λ holds if for all non-empty sets X and all functions

$$F : X^{<\lambda} \to P(X) \setminus \{\emptyset\},$$

there exists $g : \lambda \to X$ such that

$$g(\alpha) \in F(g \restriction \alpha)$$

for all $\alpha < \lambda$.

Fact

The axiom of choice is equivalent over the theory ZF *to the assertion that* DC_λ *holds for all* λ.

The axiom of choice is a global forcing axiom

Definition

Let $(P, \leq)$ be a partial order.
$D \subseteq P$ is dense if for all $q \in P$ there exists $p \leq q$ with $p \in D$.
$G \subseteq P$ is a filter if for all $q \in G$ and $r \geq q$ $r \in G$, and for all $r, s \in G$ there exists $u \leq r, s$ with $u \in G$.

Definition

Let $(P, \leq)$ be a partial order. $\mathrm{FA}_\lambda(P)$ holds if for all families $\{D_\alpha : \alpha < \lambda\}$ of dense subsets of P, there exists a filter $G \subset P$ which has non-empty intersection with all the D_α.
Let Γ be a class of partial orders. Then $\mathrm{FA}_\lambda(\Gamma)$ holds if $\mathrm{FA}_\lambda(P)$ holds for all $P \in \Gamma$.

Fact

$\mathrm{DC}_{\aleph_0}$ *is equivalent over the theory* ZF *to the assertion that* $\mathrm{FA}_{\aleph_0}(P)$ *holds for all partial orders* P.

The axiom of choice is a global forcing axiom.

Sketch of proof. I show just the direction I want to bring forward ($\forall P\ \mathsf{FA}_{\aleph_0}(P) \to \mathsf{DC}_{\aleph_0}$):

Assume $F : X^{<\omega} \to P(X) \setminus \{\emptyset\}$ is a function. Let T be the subtree of $X^{<\omega}$ given by finite sequences $s \in X^{<\omega}$ such that $s(i) \in F(s \restriction i)$ for all $i < |s|$.
Consider on the tree partial order $(T, >_T)$ the family given by the dense sets

$$D_n = \{s \in T : |s| > n\}.$$

If G is a filter on T meeting the dense sets of this family, $\cup G$ works.

The axiom of choice is a global forcing axiom.

More generally:

Definition

A partial order P is $< \lambda$-closed if all downward directed chains in P of length less than λ have a lower bound.

Let $\mathsf{AC} \restriction \lambda$ abbreviate DC_γ *holds for all* $\gamma < \lambda$ and Γ_λ be the class of $< \lambda$-closed posets.

Fact

DC_λ *is equivalent to* $\mathsf{FA}_\lambda(\Gamma_\lambda)$ *over the theory* $\mathsf{ZF} + \mathsf{AC} \restriction \lambda$.

The axiom of choice is a global forcing axiom.

Conclusion:

Fact

The axiom of choice is equivalent over the theory ZF *to the assertion that* $\mathsf{FA}_\lambda(\Gamma_\lambda)$ *holds for all cardinals* λ *(where* Γ_λ *is the class of* $< \lambda$*-closed partial orders).*

Baire's category theorem

The following is not the most general topological formulation of BCT but in essence any other form of the Baire's category theorem can be deduced from this one.

Theorem

Assume (X, τ) is a compact Hausdorff space. Let $\{D_n : n \in \mathbb{N}\}$ be a family of dense open subsets of X. Then $\bigcap_{n\in\mathbb{N}} D_n$ is dense in X.

Proof.

(By means of $\mathrm{FA}_{\aleph_0}(\Omega)$)
Consider the partial order $(P, \leq) = (\tau \setminus \{\emptyset\}, \subseteq)$. The sets

$$E_n = \{A \in \tau : \overline{A} \subseteq D_n\}$$

are open dense in P. By $\mathrm{FA}_{\aleph_0}(P)$ we can find a filter $G \subseteq P$ such that $G \cap E_n \neq \emptyset$. Let $A_n \in G \cap E_n$. Then

$$\bigcap_{n \in \mathbb{N}} D_n \supseteq \bigcap_{n \in \mathbb{N}} \overline{A_n} \neq \emptyset$$

by compactness and by definition of $A_n \in E_n$, since the family of closed sets $\{\overline{A_n} : n \in \mathbb{N}\}$ has the finite intersection property. $\square$

Actually $\mathrm{FA}_{\aleph_0}(\Gamma_{\aleph_0})$ (or $\mathrm{DC}_{\aleph_0}$) is equivalent to the conclusion of Baire's category theorem.

Baire's category theorem

More generally we have the following:

Theorem

Assume λ is a cardinal Γ is a class of partial orders such that whenever $P \in \Gamma$, its boolean completion $\mathrm{RO}(P)$ is also in Γ. TFAE:

1. $\mathrm{FA}_\lambda(\Gamma)$ *(i.e. $\mathrm{FA}_\lambda(P)$ holds for all $P \in \Gamma$).*
2. $\mathrm{FA}_\lambda(\mathrm{B})$ *holds for all complete boolean algebras* $\mathrm{B} \in \Gamma$.
3. *Whenever (X, τ) is a compact Hausdorff space such that $(\tau \setminus \{\emptyset\}, \subseteq) \in \Gamma$ and $\{D_i : i < \lambda\}$ is a family of dense open subsets of X, we have that*
$$\bigcap_{i<\lambda} D_i \neq \emptyset.$$

The equivalence of the first two items is provable in ZF. The equivalence of the third item with the first two requires the *prime ideal theorem* in order to implement *Stone's duality* between compact 0-dimensional spaces and boolean algebras.

We have already outlined the following:

> *Forcing axioms provide a natural language in which the axiom of choice and Baire's category theorem can be expressed.*

Let us now investigate further in two directions:

- How can we formulate strong forcing axioms, i.e. axioms which are really stronger than the axiom of choice?
- Can we reach maximal forcing axioms? How do we recognize a forcing axiom as maximal?

Topological formulations of forcing axioms

Let $\Omega = \Gamma_{\aleph_0}$ be the class of all posets.

$\mathsf{FA}_{\aleph_0}(\Omega)$ ($\mathsf{DC}_{\aleph_0}$) is a weak form of the axiom of choice and an equivalent formulation of Baire's category theorem.

The first direction to pursue is to step up and analyze what kind of posets P can satisfy $\mathsf{FA}_{\aleph_1}(P)$.

- AC implies that all $< \aleph_1$-closed posets P satisfy $\mathsf{FA}_{\aleph_1}(P)$.
- It is known that there are P such that $\mathsf{FA}_{\aleph_1}(P)$ provably fails (in ZF).

One such example is given by $(P, \leq)$ being the family of non-empty regular open subsets of $X^{\mathbb{N}}$ where X is uncountable and endowed with the discrete topology (i.e. $P \cong \mathit{Coll}(\omega, X)$).

Shelah has isolated a condition (that of being a *locally* SSP-*partial order*) such that:

1. (Shelah) Assume P is not SSP. Then $\mathsf{FA}_{\aleph_1}(P)$ must fail.
2. (Foreman, Magidor, Shelah) Assume ZFC + *there exists a supercompact cardinal* is consistent. Then also the theory

$$\mathsf{ZFC} + [\forall P\, (\mathsf{FA}_{\aleph_1}(P) \leftrightarrow P \text{ is locally SSP})]$$

   is consistent.

The latter axiom is named MM in the literature and appears to be a a maximal strengthening of $\mathsf{AC} \restriction \omega_2$.

MM is succesful in settling problems which are provably undecidable on the basis of ZFC alone, a sample of the consequences of MM are the following:

- $2^{\aleph_0} = \aleph_2$ and the singular cardinal hypothesis SCH holds (Foreman-Magidor-Shelah),
- There are no Kurepa trees and no Suslin trees of height ω_1 (Todorčevič?, Solovay-Tennenbaum)
- There are five uncountable linear orders such that any other uncountable linear order contains an isomorphic copy of one of those five (Moore),
- All automorphisms of the Calkin algebra are inner (Farah),
- There is a non-free Whitehead group (Shelah),
- ...

But we can go further to strenghten forcing axioms like MM and also to give an explanation grounded on logical arguments which explains its success.

Forcing axioms as density properties of class partial orders.

Definition

Let Γ be a class of complete boolean algebras and Θ be a class of complete homomorphisms between elements of Γ and closed under composition and identity maps.

- $\mathsf{B} \leq_\Theta \mathsf{Q}$ if there is a complete homomorphism $i : \mathsf{B} \to \mathsf{Q}$ in Θ.
- $\mathsf{B} \leq^*_\Theta \mathsf{Q}$ if there is a complete and *injective* homomorphism $i : \mathsf{B} \to \mathsf{Q}$ in Θ.

With these definitions $(\Gamma, \leq_\Theta)$ and $(\Gamma, \leq^*_\Theta)$ are class partial orders.

Forcing axioms as density properties of class partial orders.

In particular we can now look at these class partial orders as forcing notions, and check whether they are interesting forcing notions.

The order $\leq_{\Theta}^{*}$ is the one we use to study iterated forcing and captures the notion of complete embedding for partial orders.

$\leq_{\Theta}$ has been neglected so far but is sufficient to grant that whenever $i : \mathsf{B} \to \mathsf{Q}$ witnesses $\mathsf{Q} \leq_{\Theta} \mathsf{B}$ and G is V-generic for Q, then $i^{-1}[G]$ is V-generic for B.

Forcing axioms as density properties of class partial orders.

Theorem

The following holds:

- **Woodin:** *Assume there are class many Woodin cardinals. Then the family of* **presaturated towers** *is dense in* $(\Omega, \leq_\Omega)$ *where* Ω *stands both for the class of all complete boolean algebras and for the class of all complete homomorphisms between complete boolean algebras.*
- **Woodin:** *Assume there are class many Woodin cardinals. Then Martin's maximum is equivalent to the assertion that the family of presaturated towers is dense in* $(\mathsf{SSP}, \leq_\Omega)$.
- **V.:** *Assume there are class many Woodin cardinals Then* MM^{++} *(a strong form of* MM*) is equivalent to the assertion that the family of presaturated towers is dense in* $(\mathsf{SSP}, \leq_{\mathsf{SSP}})$*,where* $\mathsf{B} \geq_{\mathsf{SSP}} \mathsf{Q}$ *iff there is* $i : \mathsf{B} \to \mathsf{Q}$ *complete homomorphism such that*

$$[\![\mathsf{Q}/i[\dot{G}_\mathsf{B}] \in \mathsf{SSP}]\!]_\mathsf{B} = 1_\mathsf{B}.$$

Maximal strengthenings of AC $\restriction \kappa$

Definition (V.)

Let Γ be a definable class of partial orders.
CFA(Γ) holds if the class of *strongly* presaturated towers is dense in $(\Gamma, \leq_\Gamma)$.

I will skip the definition of strongly presaturated tower for this talk......

Fact

CFA(SSP) $\equiv \mathsf{MM}^{+++} \to \mathsf{MM}^{++} \to \mathsf{MM}$.

Theorem (V. Jour. Amer. Math. Soc. 2015)

CFA(SSP) $\equiv \mathsf{MM}^{+++}$ *is consistent relative to the existence of an almost huge cardinal.*

For any Γ such that CFA(Γ) let

$$\kappa_\Gamma = \min\{\theta : \mathsf{FA}_\theta(\Gamma) \text{ fails}\}.$$

$\kappa_{\mathsf{SSP}} = \omega_2$.

Theorem (V. Jour. Amer. Math. Soc. 2015)

Assume Γ is a definable class of forcings closed under two step iterations. There are first order expressible properties $A(x, y)$, $B(x, y)$, $C(x, y)$, $D(x, y)$ such that whenever $A(\Gamma, \kappa)$, $B(\Gamma, \kappa)$, $C(\Gamma, \kappa)$, $D(\Gamma, \kappa)$ hold and $\phi(x)$ is a formula in one free variable, the following are equivalent for any

$$T \supseteq \mathsf{ZFC} + \mathsf{CFA}(\Gamma) + \textit{there are class many reflecting cardinals}:$$

1. $T \vdash p \subseteq \kappa \wedge \phi^{L(ON^{<\kappa})}(p)$,
2. *T proves that for some* $\mathsf{B} \in \Gamma$

$$[\![p \subseteq \kappa \wedge \phi^{L(ON^{<\kappa})}(p)]\!]_{\mathsf{B}} = 1_{\mathsf{B}}.$$

Theorem (V.)

Assume Γ *is a definable class of forcings closed under two step iterations. Assume* $A(\Gamma,\kappa)$, $B(\Gamma,\kappa)$, $C(\Gamma,\kappa)$, $D(\Gamma,\kappa)$ *hold. Then* $\mathsf{CFA}(\Gamma)$ *is consistent relative to the existence of an almost huge cardinal and* $\mathsf{CFA}(\Gamma) \rightarrow \mathsf{FA}_{<\kappa}(\Gamma)$.

Loosely speaking

- $A(\Gamma, \kappa)$ holds if one has an iteration theorem which preserves property Γ through limit stages of an iteration (semiproperness, properness, CCC, axiom A are all examples of Γ for which $A(\Gamma, \omega_2)$ holds).
- $B(\Gamma, \kappa)$ requires that Γ has a simple definition in terms of its logical complexity (semiproperness, properness, CCC, axiom A are all examples of Γ for which $B(\Gamma, \omega_2)$ holds).
- $C(\Gamma, \kappa)$ requires that $(\Gamma, \leq_\Gamma)$ has a dense suborder D with the following property for all $\mathsf{B}, \mathsf{C} \in D$:

  $\mathsf{B} \leq_\Gamma \mathsf{C}$ *if and only if there is just one* $i : \mathsf{B} \to \mathsf{C}$ *witnessing it.*

- $D(\Gamma, \kappa)$ requires that all $< \kappa$-closed partial orders are in Γ.

More precisely:

Definition (V. Asperò)

Let Γ be a definable class of partial orders closed under two steps iterations.

1. $A(\Gamma, \kappa)$ holds if $(\Gamma, \leq^*_\Gamma)$ is closed under set sized descending sequences and there is a fixed cardinal $\eta < \kappa$ such that for all $\leq^*_\Gamma$-descending sequences

$$\mathcal{F} = \{B_\alpha : \alpha < \eta\}$$

$\varinjlim(\mathcal{F}) \in \Gamma$ is a lower bound for $\mathcal{F}$ in $\leq^*_\Gamma$.

2. $B(\Gamma, \kappa)$ holds if there is a Δ_0-formula $\phi(x, y, A)$ with $A \subseteq \kappa$ such that for all $S \in V$ and $\mathsf{B} \in \Gamma$

$$V \models \forall x \phi(x, S, A) \leftrightarrow V^{\mathsf{B}} \models \forall x \phi(x, S, A).$$

For example the formula

$\phi(f, S, \aleph_0) \equiv S \subseteq [\cup S]^{\aleph_0}$ *and* $f : (\cup S)^{<\omega} \to \cup S$ *and there is* $X \in S$ *such that* $f[X^{<\omega}] \subseteq X$.

is Σ_0 and is used to define properness and witnesses $B(\text{Proper}, \omega_2)$.

Theorem (Aspero)

Properties $A(\Gamma, \omega_2)$, $B(\Gamma, \omega_2)$, $C(\Gamma, \omega_2)$, $D(\Gamma, \omega_2)$ *hold for any* Γ *given by the intersection of a class in* A_0 *and another class in* A_1 *where*

- $A_0 = \{$*proper, semiproper*$\}$,
- $A_1 = \{$*all,* ω^ω*-bounding, preserving a fixed Suslin tree* $T\}$.

There is a ninth class Γ^* *such that* $A(\Gamma^*, \omega_2)$, $B(\Gamma^*, \omega_2)$, $C(\Gamma^*, \omega_2)$, $D(\Gamma, \omega_2)$ *hold,* CFA(Γ^*) *is consistent and it implies* CH.

The major hurdle here is to find an interesting class Γ such that $A(\Gamma, \omega_3), B(\Gamma, \omega_3), C(\Gamma, \omega_3)$ can hold at Γ with $\kappa_\Gamma > \aleph_2$.

Intermezzo – Why I like the category forcings $(\Gamma, \leq_\Gamma)$:

These categories have many surprising and nice features:

Theorem (V.)

Assume that $\delta > \kappa$ is supercompact and $A(\Gamma,\kappa)$, $B(\Gamma,\kappa)$, $C(\Gamma,\kappa)$, $D(\Gamma,\kappa)$ hold. Then $(\Gamma \cap V_\delta, \leq_\Gamma \restriction V_\delta)$ is a partial order $\mathsf{U}^\Gamma_\delta \in \Gamma$.
Moreover:

- $\mathsf{B} \geq_\Gamma \mathsf{U}^\Gamma_\delta \restriction \mathsf{B}$ *for all* $\mathsf{B} \in \Gamma \cap V_\delta$.
- U^Γ_δ *forces* $\mathsf{FA}_{<\kappa}(\Gamma)$.

Theorem (V.)

Assume δ is a reflecting cardinal and CFA(Γ) *holds (i.e. there are densely many* **strongly presaturated towers** *in* $(\Gamma, \leq_\Gamma)$*).*
Then U^Γ_δ is itself a strongly presaturated tower.

In general the following holds for "simple" properties $\phi(x)$:

$\phi(\mathsf{U}^\Gamma_\delta)$ *holds iff* $\{\mathsf{B} \in \mathsf{U}^\Gamma_\delta : \phi(\mathsf{B})$ *holds*$\}$ *is dense in* U^Γ_δ.

Generic absoluteness: standard formulation

Assume $T \subseteq \mathrm{ZFC}$, T_0 is a family of sentences in the language of T, Γ is a class of forcing notions.
T_0 is Γ-generically invariant for T if for all $S \supseteq T$ and any $\phi \in T_0$ any of the following three equivalent conditions holds:

1. $S \vdash \phi$
2. $S \vdash$ there is $\mathsf{B} \in \Gamma$ such that $[\![\phi]\!] = 1_{\mathsf{B}}$ and $[\![T]\!] = 1_{\mathsf{B}}$.
3. $S \vdash [\![\phi]\!] = 1_{\mathsf{B}}$ for all $\mathsf{B} \in \Gamma$ such that $[\![T]\!] = 1_{\mathsf{B}}$.

The class of boolean valued models of the form V^{B} gives a complete semantic for T_0 with respect to any $S \supseteq T$.

Theorem (Woodin)

Let T_0 be the theory with real parameters of $L[\mathrm{ON}^\omega]$ and $T = \mathrm{ZFC}+$large cardinals.
Then T_0 is Γ-generically invariant for T where Γ is the class of all posets.

Fact (Shelah)

Assume T_0 is third order number theory and $T \supseteq \mathrm{ZFC}$. Then T_0 cannot be Γ-generically invariant for T unless Γ is contained in SSP, *the class of stationary set preserving posets.*

Theorem (V. Asperó)

Let T_0 be the theory with parameters in $P(\omega_1)$ of $L[\mathrm{ON}^{\omega_1}]$. There are at least nine distinct Γ such that, letting

$$T_\Gamma = \mathrm{ZFC} + \textit{ large cardinals } + \mathrm{CFA}(\Gamma),$$

T_0 is Γ-generically invariant for T_Γ.

Forcing axioms as clopen class games: the iterated resurrection axioms

The next family of forcing axioms leading to generic absoluteness results are inspired by Hamkins and Johnstone's resurrection axioms, and by Tsaprounis elaborations on their work.

The following is the outcome of a joint work with Audrito, currently completing with me his Ph.D. thesis.

We can develop the definition of the resurrection axiom starting from a model-theoretic point of view.

Theorem

Let $M \subset N$ be models of a language $\mathcal{L}$. Then TFAE:

- *M is existentially closed in N ($M \prec_1 N$),*
- *M has* **resurrection**, *i.e. it exists a larger $M' \supseteq N$ such that $M \prec M'$*

If we restrict the above properties to models of set theory of the form $H_{\mathfrak{c}}^{M}$ (where $\mathfrak{c} = 2^{\aleph_0}$) and consider only model extensions obtained by forcing in a fixed class $\Gamma \subseteq \mathsf{SSP}$, we obtain respectively:

- M satisfies $\mathsf{BFA}_{\aleph_1}(\Gamma)$,
- M satisfies $\mathsf{RA}(\Gamma)$, the resurrection axiom by Hamkins and Johnstone.

To formalize correctly the iterated resurrection axioms it is more convenient to use the Morse-Kelley theory of sets MK.

Definition

Let M be a model of MK and Γ be a class belonging to M and closed under two-steps iterations.
RA(Γ) holds in M if and only if

> *Forall $N \supseteq M$ obtained by some forcing in Γ^M, there exists an $M' \supseteq N$ a further extension by a forcing in Γ^N such that $H_{\mathfrak{c}}^M \prec H_{\mathfrak{c}}^{M'}$.*

$$
\begin{tikzcd}
M \arrow[rr, "\Sigma_\omega(H_{\mathfrak{c}})\text{-elem.}"] \arrow[dr, "\Gamma"'] & & M' \\
& N \arrow[ur, "\Gamma"'] &
\end{tikzcd}
$$

Resurrection axioms have been introduced recently by Hamkins and Johnstone, and are interesting since they can prove some consequences of Martin's maximum, while having much lower consistency strength.

To simplify matters we concentrate just on classes $\Gamma \subseteq \mathrm{SSP}$ and we fix the cardinal $\kappa = \mathfrak{c} = 2^{\aleph_0} \leq \aleph_2$. However suitably reformulated our results are modular in κ.

Definition

Let V be a a model of MK and $\Gamma \in V$ be any definable class of partial orders closed under two-steps iterations.

The Γ-resurrection game $\mathcal{G}^{\mathrm{RA}}$ is as follows:

Player I (*Kill*) plays couples (α_n, B_n) where α_n is an ordinal such that $\alpha_{n+1} < \alpha_n$ and $\mathrm{B}_n \in \Gamma$ is such that $\mathrm{B}_{n+1} \leq_\Gamma \mathrm{C}_n$.

Player II (*Resurrect*) plays boolean algebras C_n such that $H_{\mathfrak{c}}^{\mathrm{C}_n} \prec H_{\mathfrak{c}}^{\mathrm{C}_{n+1}}$ and $\mathrm{C}_n \leq_\Gamma \mathrm{B}_n$.

The last player who can move wins.

$$
\begin{tikzcd}
\text{I } (\textit{Kill}). & (\alpha_0, \mathrm{B}_0) \arrow[dr] & & (\alpha_1, \mathrm{B}_1) \arrow[dr] & & (\alpha_2, \mathrm{B}_2) \arrow[dr, dashed] & \\
\text{II } (\textit{Resurrect}). & & \mathrm{C}_0 \arrow[ur] & & \mathrm{C}_1 \arrow[ur] & & \dots
\end{tikzcd}
$$

This game is formalizable in MK and is a clopen class game and thus determined.

Definition

Let V be a a model of MK and $\Gamma \in V$ be any definable class of partial order closed under two-steps iterations.

$\mathsf{RA}_{\mathsf{ON}}(\Gamma)$ holds in V if player II has a winning strategy in the game $\mathcal{G}^{\mathsf{RA}}$.

$\mathsf{RA}_{\alpha}(\Gamma)$ holds in V if player II has a winning strategy in the game $\mathcal{G}^{\mathsf{RA}}$ where I starts with (α_0, B_0) for some $\alpha_0 < \alpha$.

The $\mathsf{RA}_{\mathsf{ON}}(\Gamma)$ and $\mathsf{RA}_{\alpha}(\Gamma)$ are first order expressible in models of MK.

The iterated resurrection axioms can also be phrased as a density property:

Lemma

Assume V is a model of MK *and* Γ *is definable in* V. *Then* $\mathrm{RA}_\alpha(\Gamma)$ *holds in* V *if and only if for all* $\beta < \alpha$ *the class*

$$\left\{\mathrm{B} \in \Gamma : \; H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V^{\mathrm{B}}} \wedge V^{\mathrm{B}} \models \mathrm{RA}_\beta(\Gamma)\right\}$$

is dense in $(\Gamma, \leq_\Gamma)$.

From this strengthened axiom we can obtain:

Theorem (Audrito, V.)

$\mathsf{MK} + \mathsf{RA}_\omega(\Gamma)$ *has generic absoluteness for* Θ *the formulas relativized to* $H_{\mathfrak{c}}$ *and forcing in* Γ*.*

With respect to the previous results about $\mathsf{ZFC} + \mathsf{CFA}(\Gamma) + \mathsf{LC}$:

- Θ is smaller since in general $\mathfrak{c} = \omega_2$ and $H_{\mathfrak{c}} \subset L([\mathsf{ON}]^{\omega_1})$,
- it is more general since it holds for many more Γ (essentially to run the machinery we just need Γ to satisfy $A(\Gamma, \kappa)$ and $B(\Gamma, \kappa)$ *(but not* $C(\Gamma, \kappa)$*,* $D(\Gamma, \kappa)$*)*,
- it has lower consistency strength then $\mathsf{CFA}(\Gamma)$ for many Γ.

Lemma

$\mathsf{MK} + \mathsf{RA}_n(\Gamma)$ *has generic absoluteness for* Θ *the* Σ_{n+1} *formulas relativized to* $H_{\mathfrak{c}}$ *and forcing in* Γ.

Proof.

By induction on n, consider a Σ_{n+1} formula $\phi = \exists x \psi(x)$ and draw the following:

$$\begin{tikzcd} \underline{\mathsf{RA}_n(\Gamma)} & H_{\mathfrak{c}}^M \arrow[rr, "\Sigma_\omega"] \arrow[dr, "\Sigma_n"'] & & H_{\mathfrak{c}}^{M'} & \underline{\mathsf{RA}_{n-1}(\Gamma)} \\ & & H_{\mathfrak{c}}^N \arrow[ur, "\Sigma_n"'] & & \\ & & \underline{\mathsf{RA}_n(\Gamma)} & & \end{tikzcd}$$

where N is the generic extension of M by a forcing in Γ.

- $M \models \psi^{H_{\mathfrak{c}}}(a) \Rightarrow N \models \psi^{H_{\mathfrak{c}}}(a)$ so $M \models \exists x \psi^{H_{\mathfrak{c}}}(x) \Rightarrow N \models \exists x \psi^{H_{\mathfrak{c}}}(x)$,
- $N \models \exists x \psi^{H_{\mathfrak{c}}}(x) \Rightarrow M' \models \exists x \psi^{H_{\mathfrak{c}}}(x)$ (same argument) $\Rightarrow M \models \exists x \psi^{H_{\mathfrak{c}}}(x)$ (elementarity).

$\square$

Theorem (Audrito, V.)

The following holds:

- $\mathsf{RA}_{\mathsf{ON}}(\Gamma)$ *for* Γ *satisfying* $A(\Gamma, \omega_2)$ *is consistent relative to the existence of a Mahlo cardinal over the theory* MK,
- $\mathsf{RA}_{\mathsf{ON}}(\mathsf{SSP})$ *is consistent relative to a stationary limit of supercompact cardinals over the theory* MK,
- $\mathsf{MM}^{+++} \equiv \mathsf{CFA}(\mathsf{SSP}) \Rightarrow \mathsf{RA}_{\mathsf{ON}}(\mathsf{SSP})$.

Observe that κ is measurable is equivalente to the following form of forcing axiom:

> *The boolean algebra $P(\kappa)/I$ contains a ultrafilter meeting all maximal antichains of size less than κ (where I is the ideal given by bounded subsets of κ).*

We can elaborate further in order to express several other large cardinal axioms as suitable forcing axioms.

For examples to express supercompactness of κ it is less straightforward to express for which families of predense sets on certain boolean algebras of the form $P(P(\kappa))/I$ we aim to find a generic ultrafilter.......

In conclusion:

- Forcing axioms are natural generalizations of AC and BCT,
- There are diverse approaches which transfer ideas and methods arising in other contexts and produce strong forms of forcing axioms which yield generic absoluteness results.
- Generic absoluteness transform forcing from a tool to prove independence results in a tool to prove results (i.e. theorems).
- A major mathematical hurdle to further expand on this approach is to find the righ iteration theorem for forcings preserving $\aleph_1$ and $\aleph_2$.
- Another (probably less demanding) line of investigation is to understand what is the correct formulation in terms of forcing axioms of several large cardinal axioms.

Thanks for your attention!

Bibliography I

Giorgio Audrito and Matteo Viale.
Absoluteness via resurrection.
arXiv:1404.2111 (submitted), 2014.

Matteo Viale.
Category forcings, MM^{+++}, and generic absoluteness for the theory of strong forcing axioms.
Journal of the American Mathematical Society, published electronically: August 19, 2015.