

Potentialism about set theory

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But [the set-theoretic paradoxes] are only apparent 'contradictions', and depend solely on confusing set theory itself, which is not categorically determined by its axioms, with individual models representing it. What appears as an 'ultrafinite non- or super-set' in one model is, in the succeeding model, a perfectly good, valid set with both a cardinal number and an ordinal type, and is itself a foundation stone for the construction of a new domain. (Zermelo, 1930)

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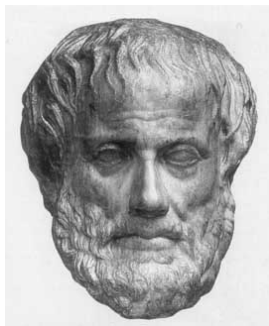
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- Show that both concepts have substantial explanatory value
- Explore logical consequences of adopting the successor concepts

Aristotle's notion of potential infinity (I)



For generally the infinite is as follows: there is always another and another to be taken. And the thing taken will always be finite, but always different (Physics, 206a27-29).

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Contrast the notion of actual or *completed* infinity:

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Potentiality in set theory

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Contrasts with potentialism about arithmetic

- While potential infinity is concerned with ω , potentialism about sets is concerned with Ω .
- While completion of the natural numbers is consistent, completion of the sets is not.
- Aristotle's modality is metaphysical. Not so in the case of potentialism about sets.

A three-way distinction

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Liberal potentialism: Mathematical objects are generated successively. It is impossible to complete the process of generation.

Strict potentialism: Additionally, every truth is 'made true' at some stage of the generative process.

The notion of incompleteness

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This formalization presupposes the *stability* of \prec :

- $x \prec yy \rightarrow \Box(x \prec yy)$
- $x \not\prec yy \rightarrow \Box(x \not\prec yy)$

Cantor on completability and set formation

[We must] distinguish two kinds of multiplicities [...] For a multiplicity can be such that the assumption that all of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call absolutely infinite or inconsistent multiplicities. [...] If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into 'one thing', I call it a consistent multiplicity or a 'set'. (1899 letter to Dedekind)

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- Consistent/inconsistent multiplicity \approx completable/incompletable condition
- Consistent multiplicities form sets *because* they are completable.

Infinity vs. incompleteness

<i>our terminology</i>	<i>Aristotle</i>	<i>today</i>	<i>Cantor</i>
infinite and incomplete	infinite	X	absolutely infinite
infinite and completable	X	infinite	transfinite
finite (and completable)	finite	finite	finite

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My view

- Incompleteness generalizes the ancient notion of potential infinity.
- We should follow Cantor, not Aristotle, on the completeness of the natural numbers.
- Incompleteness provides a useful *supplement* to the modern notion of infinity.

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2. *We need a bridge between the non-modal language of ordinary mathematics and the modal notion of incompleteness*

The Gödel translation (I)

The non-trivial clauses of the translation G are:

$$\phi \mapsto \Box\phi \quad \text{for } \phi \text{ atomic}$$

$$\neg\phi \mapsto \Box\neg\phi^G$$

$$\phi \rightarrow \psi \mapsto \Box(\phi^G \rightarrow \psi^G)$$

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Theorem (Intuitionistic mirroring)

Let \vdash_{int} be intuitionistic first-order deducibility. Let \vdash_{S4} be deducibility in classical first-order logic plus S4. Then we have:

$$\phi_1, \dots, \phi_n \vdash_{int} \psi \quad \text{iff} \quad \phi_1^G, \dots, \phi_n^G \vdash_{S4} \psi^G.$$

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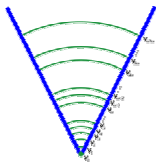
Consider this axiom of Peano and Heyting arithmetic:

$$\forall m \exists n \text{SUCCESSOR}(m, n) \quad (1)$$

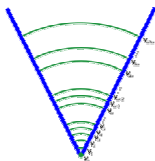
Its translation requires that each world that contains one number, contains all of them!

$$\Box \forall m \exists n \text{SUCCESSOR}(m, n) \quad (2)$$

The potentialist translation (I)

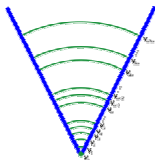


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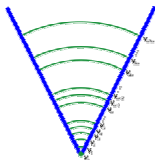
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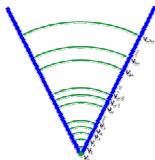
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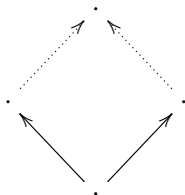
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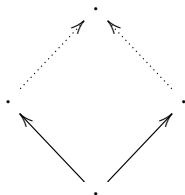
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So we adopt $S4.2 = S4 + G$.

The potentialist translation (III)

Theorem (Classical mirroring)

Let \vdash^\diamond be provability by \vdash , S4.2, and axioms stating that every atomic predicate is rigid, but with no higher-order comprehension. Then we have:

$$\phi_1, \dots, \phi_n \vdash \psi \quad \text{iff} \quad \phi_1^\diamond, \dots, \phi_n^\diamond \vdash^\diamond \psi^\diamond.$$

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Upshot: liberal potentialists are entitled to classical (first-order) logic.

Plural logic in the context of incompleteness

Question: When does a condition ϕ define a plurality:

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Answer: Every plurality is exhausted by some world.

The availability of classical quantification

Classical quantification functions like (perhaps infinite) conjunctions or disjunctions of instances:

$$(\forall x : \phi(x))\psi(x) \leftrightarrow \bigwedge_{\phi(\bar{a})} \psi(\bar{a}) \quad (3)$$

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A strict potentialist denies that the sets are traversable—although any one set is. A generalization over all sets cannot be ‘made true’ by sets not yet formed.

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But this is problematic, both in its own right, and especially in connection with set theory.

Alternatives to classical quantification (II)

Is there a natural number that has some decidable property P ?

Only the finding that has actually occurred of a determinate number with the property P can give a justification for the answer “Yes,” and—since I cannot run a test through all numbers—only the insight, that it lies in the essence of number to have the property P , can give a justification for the answer “No”; Even for God no other ground for decision is available. But these two possibilities do not stand to one another as assertion and negation. (Weyl, 1921, p. 97)

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This is an intensional conception of generality which is

- independent of radical anti-realism
- available to the strict potentialist
- interestingly modelled by Kleene realizability

A bimodal semantics (I)

At each stage of the generative process, we have

- resources to generate more objects, i.e. possibilities are ruled in: G-accessibility
- resources to establish constraints on all further generation, i.e. possibilities are ruled out: D-accessibility

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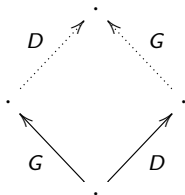
As before, G-modality is subject to S4.2, and D-modality, to S4.

How do the two modalities relate to one another? Since $\leq_G \subseteq \leq_D$, we firstly adopt the axiom:

$$\Box_D \phi \rightarrow \Box_G \phi. \quad (\text{Incl})$$

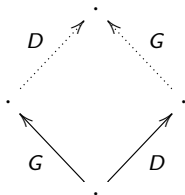
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So we finally adopt a 'mixed' version of G:

$$\diamond_G \Box_D \phi \rightarrow \Box_D \diamond_G \phi \quad (\text{Mixed-G})$$

Intuitionistic bimodal mirroring

The strict potentialist translates:

$$\begin{aligned}\phi &\mapsto \Box_D \neg \phi^* \\ \phi \rightarrow \psi &\mapsto \Box_D (\phi^* \rightarrow \psi^*) \\ \forall x \phi &\mapsto \Box_D \forall x \phi^* \\ \exists x \phi &\mapsto \Diamond_G \exists x \phi^*\end{aligned}$$

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Theorem (Intuitionistic bimodal mirroring)

Let \vdash_{int} be intuitionistic deducibility but with any higher-order comprehension axioms removed. Let \vdash^* be the corresponding deducibility relation in the mentioned (classical) bimodal system, along with axioms asserting the G -stability of each atomic predicate. Then, for any non-modal formulas $\phi_1, \dots, \phi_n, \psi$, we we have:

$$\phi_1, \dots, \phi_n \vdash_{int} \psi \quad \text{iff} \quad \phi_1^*, \dots, \phi_n^* \vdash^* \psi^*.$$

Concluding remarks

Recall our three players

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Logical manifestations

- incompleteness: restrict plural comprehension
- intraversability: semi-intuitionistic logic (i.e. global quantification is intuitionistic, bounded is classical)
- Exercise: what does intraversability mean for intensional second-order comprehension?

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