Potentialism about set theory

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Open-endedness in set theory

But [the set-theoretic paradoxes] are only apparent 'contradictions', and depend solely on confusing set theory itself , which is not categorically determined by its axioms, with individual models representing it. What appears as an 'ultrafinite non- or super-set' in one model is, in the succeeding model, a perfectly good, valid set with both a cardinal number and an ordinal type, and is itself a foundation stone for the construction of a new domain. (Zermelo, 1930)

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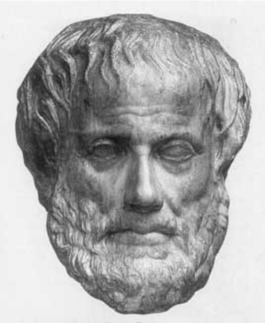
- Show that both concepts have substantial explanatory value
- Explore logical consequences of adopting the successor concepts

Aristotle's notion of potential infinity (I)



For generally the infinite is as follows: there is always another and another to be taken. And the thing taken will always be finite, but always different (Physics, 206a27-29).

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Contrast the notion of actual or *completed* infinity:

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Contrasts with potentialism about arithmetic

- While potential infinity is concerned with ω, potentialism about sets is concerned with Ω.
- While completion of the natural numbers is consistent, completion of the sets is not.
- Aristotle's modality is metaphysical. Not so in the case of potentialism about sets.

A three-way distinction

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Liberal potentialism: Mathematical objects are generated successively. It is impossible to complete the process of generation.

Strict potentialism: Additionally, every truth is 'made true' at some stage of the generative process.

The notion of incompletability

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This formalization presupposes the *stability* of $\prec$:

- $x \prec yy \rightarrow \Box(x \prec yy)$
- $x \not\prec yy \rightarrow \Box(x \not\prec yy)$

Cantor on completability and set formation

[We must] distinguish two kinds of multiplicities [. . .]For a multiplicity can be such that the assumption that all of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call absolutely infinite or inconsistent multiplicities. [. . .] If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into 'one thing', I call it a consistent multiplicity or a 'set'. (1899 letter to Dedekind)

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- Consistent/inconsistent multiplicity $\approx$ completable/incompletable condition
- Consistent multiplicities form sets *because* they are completable.

Infinity vs. incompletability

our terminology	*Aristotle*	*today*	*Cantor*
infinite and incompletable	infinite	✗	absolutely infinite
infinite and completable	✗	infinite	transfinite
finite (and completable)	finite	finite	finite

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- Incompletability generalizes the ancient notion of potential infinity.

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My view

- Incompletability generalizes the ancient notion of potential infinity.
- We should follow Cantor, not Aristotle, on the completability of the natural numbers.
- Incompletability provides a useful *supplement* to the modern notion of infinity.

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2. We need a bridge between the non-modal language of ordinary mathematics and the modal notion of incompletability

The Gödel translation (I)

The non-trivial clauses of the translation G are:

$$\begin{aligned} \phi &\mapsto \Box\phi \qquad \text{for } \phi \text{ atomic} \\ \neg\phi &\mapsto \Box\neg\phi^G \\ \phi \to \psi &\mapsto \Box(\phi^G \to \psi^G) \\ \forall x\,\phi &\mapsto \Box\forall x\,\phi^G \end{aligned}$$

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Theorem (Intuitionistic mirroring)

Let $\vdash_{int}$ be intuitionistic first-order deducibility. Let $\vdash_{S4}$ be deducibility in classical first-order logic plus S4. Then we have:

$$\phi_1, \ldots, \phi_n \vdash_{int} \psi \quad \textit{iff} \quad \phi_1^G, \ldots, \phi_n^G \vdash_{S4} \psi^G.$$

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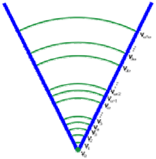
Consider this axiom of Peano and Heyting arithmetic:

$$\forall m \exists n \, \textsc{Successor}(m, n) \tag{1}$$

Its translation requires that each world that contains one number, contains all of them!

$$\Box \forall m \exists n \, \textsc{Successor}(m, n) \tag{2}$$

The potentialist translation (I)



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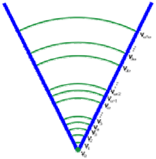

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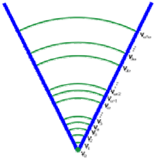

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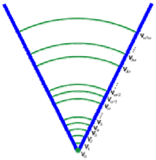

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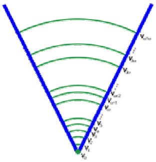

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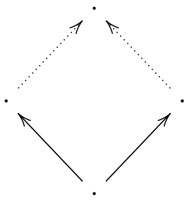

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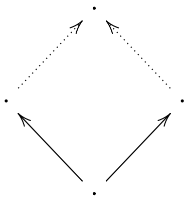

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So we adopt S4.2 = S4 + G.

The potentialist translation (III)

Theorem (Classical mirroring)

Let $\vdash^{\Diamond}$ be provability by $\vdash$, S4.2, and axioms stating that every atomic predicate is rigid, but with no higher-order comprehension. Then we have:

$$\phi_1, \ldots, \phi_n \vdash \psi \quad \textit{iff} \quad \phi_1^{\Diamond}, \ldots, \phi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond}.$$

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Upshot: liberal potentialists are entitled to classical (first-order) logic.

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Answer: Every plurality is exhausted by some world.

The availability of classical quantification

Classical quantification functions like (perhaps infinite) conjunctions or disjunctions of instances:

$$(\forall x : \phi(x))\psi(x) \leftrightarrow \bigwedge_{\phi(\bar{a})} \psi(\bar{a}) \tag{3}$$

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A strict potentialist denies that the sets are traversable—although any one set is. A generalization over all sets cannot be 'made true' by sets not yet formed.

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But this is problematic, both in its own right, and especially in connection with set theory.

Alternatives to classical quantification (II)

Is there a natural number that has some decidable property P?

> *Only the finding that has actually occurred of a determinate number with the property P can give a justification for the answer "Yes," and—since I cannot run a test through all numbers—only the insight, that it lies in the essence of number to have the property P, can give a justification for the answer "No"; Even for God no other ground for decision is available. But these two possibilities do not stand to one another as assertion and negation. (Weyl, 1921, p. 97)*

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This is an intensional conception of generality which is

- independent of radical anti-realism
- available to the strict potentialist
- interestingly modelled by Kleene realizability

A bimodal semantics (I)

At each stage of the generative process, we have

- resources to generate more objects, i.e. possibilities are ruled in: G-accessibility
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How do the two modalities relate to one another? Since $\leq_G \subseteq \leq_D$, we firstly adopt the axiom:

$$\Box_D \phi \rightarrow \Box_G \phi. \tag{Incl}$$

A bimodal semantics (II)

We adopt a 'mixed' directedness property:

$$\begin{tikzcd} & \cdot & \\ \cdot \arrow[ur, dotted, "D"] & & \cdot \arrow[ul, dotted, "G"'] \\ & \cdot \arrow[ul, "G"] \arrow[ur, "D"'] & \end{tikzcd}$$

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So we finally adopt a 'mixed' version of G:

$$\Diamond_G \Box_D \phi \rightarrow \Box_D \Diamond_G \phi \tag{Mixed-G}$$

Intuitionistic bimodal mirroring

The strict potentialist translates:

$$\phi \mapsto \Box_D \neg \phi^*$$
$$\phi \to \psi \mapsto \Box_D(\phi^* \to \psi^*)$$
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Theorem (Intuitionistic bimodal mirroring)

Let $\vdash_{int}$ be intuitionistic deducibility but with any higher-order comprehension axioms removed. Let $\vdash^*$ be the corresponding deducibility relation in the mentioned (classical) bimodal system, along with axioms asserting the G-stability of each atomic predicate. Then, for any non-modal formulas $\phi_1, \ldots, \phi_n, \psi$, we we have:

$$\phi_1, \ldots, \phi_n \vdash_{int} \psi \quad \textit{iff} \quad \phi_1^*, \ldots, \phi_n^* \vdash^* \psi^*.$$

Concluding remarks

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Two successor concepts to potential infinity

- incompletability: there is no stage at which all ϕ’s are available
- intraversability: classical quantification is unavailable

Concluding remarks

Recall our three players

- actualism: fully static picture
- liberal potentialism: objects are generated
- strict potentialism: additionally, every truth is ‘made true’ at some stage

Two successor concepts to potential infinity

- incompletability: there is no stage at which all ϕ’s are available
- intraversability: classical quantification is unavailable

Logical manifestations

- incompletability: restrict plural comprehension
- intraversability: semi-intuitionistic logic (i.e. global quantification is intuitionistic, bounded is classical)
- Exercise: what does intraversability mean for intensional second-order comprehension?

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