Is There a Good Argument for Mathematical Indeterminacy?

Jared Warren and Daniel Waxman
New York University
SoTFoM III + Hyp, KGRC, Vienna, September 21st, 2015

1 Introduction

Contemporary mathematics is replete with independence results. Many have taken these results to bear, somehow, on the case that mathematics is indeterminate. We're interested in the following questions:

1. How do independence results in mathematics bear on the question of determinacy? and
2. Is there a good argument from independence to indeterminacy? If so, how does it run?

A preliminary distinction: questions of determinacy are sometimes taken to be equivalent to questions of pluralism, but we think this is a mistake. Here's how we'll use the relevant terminology:

Pluralism there are apparently incompatible theories of D that are all equally correct.

Making this clear would involve saying more about what is meant by "apparently incompatible", "a theory of D" and by "equally correct". We won't try to do this here. Our concern is rather with:

Indeterminacy at least some D claims do not have a determinate truth value, i.e. are not determinately true or determinately false.

The notion of determinacy here is much used but somewhat under-discussed. We'll take the view that the relevant notion is roughly equivalent to "true on any admissible interpretation of the language" and that there's a story to tell about an associated cognitive role (so that, for instance, classifying some claim as indeterminate precludes further rational inquiry into its truth value). In particular we want to deny that indeterminacy is merely epistemic, as an expression of inexorable ignorance or the like.

2 From Independence to Indeterminacy?

Here is a simple argument from independence to indeterminacy, using CH and ZFC as our example.

A1 $ZFC \nvdash CH$ and $ZFC \nvdash \neg CH$, i.e. CH is independent of ZFC;

A2 If a sentence in the language of set theory is independent of ZFC, then it is indeterminate;

A3 *So*: CH is indeterminate (1,2)

Premise 2 expresses a simple – many would say overly simplistic – picture of mathematical truth, according to which a sentence in the language of set theory is determinately true just in case it is entailed by a particular, distinguished axiomatic theory of sets (ZFC).

An analogous account of arithmetical truth leads to the following argument, using the Rosser sentence for PA:

B1 $PA \nvdash R$ and $PA \nvdash \neg R$, so R is independent of PA;

B2 If an arithmetical sentence is independent of PA, then it is indeterminate;

B3 *So*: R is indeterminate (1,2)

Problem: the conclusion of *this* argument is very hard to swallow. Most philosophers and mathematicians would reject it, because there seem to be good reasons for thinking that the Rosser sentence for PA is determinately true. If that's right, then there's severe pressure to reject the picture of mathematical truth underlying both of these arguments.

3 The Metasemantic Challenge

According to the simplistic view above, the determinate arithmetical truths are precisely those provable in PA. (This is presumably meant to be an explanatory, not merely an extentional, alignment). An account of this kind is a metasemantic view: whereas semantics concerns issues like truth, reference, meaning, and other such properties, metasemantics is about the nature of these properties, the facts in virtue of which they arise.

So what we really need, in order to assess arguments from independence to indeterminacy, is to delve deeper into issues surrounding the metasemantics of mathematical theories.

This gives rise to what we'll call *the metasemantic challenge,* a general challenge faced by views in the philosophy of mathematics:

The Metasemantic Challenge To the extent that you think that mathematical (arithmetical, set theoretic, etc) claims are determinate, you face a theoretical obligation to explain how that determinacy arises.

There's an analogy to the (Benacerraf-Field) epistemological challenge – viz., roughly: to the extent that you think that our mathematical beliefs are reliable, you face a theoretical obligation to explain how that reliability arises. You might think that the metasemantic challenge is a form of scepticism, or an imposition of "first philosophy" but this would be a mistake.

With the challenge on the table, the obvious question arises: are there views that are destablized by the metasemantic challenge (analogous to the way in which, according to many, the epistemological challenge destabilizes platonist views of a certain kind)?

4 Constraints on an Acceptable Metasemantics

We think there's at least a strong *prima facie* case to be made that any acceptable overall metasemantic account must satisfy the following constraints:

The Metaphysical Constraint Mathematical objects cannot play an ineliminable role in our best metasemantic account of mathematical language.

Distinguish between different "grades of platonic involvement".

The first grade: admitting the existence of real, mind-independent mathematical objects. (Of course, many will balk even here.)

The second grade: assigning these objects a serious or ineliminable explanatory role in our theorising – including theorising about metasemantics.

The metaphysical constraint follows immediately from rejection of the second grade of involvement. Two reasons to reject the second grade of involvement:

(1) Something like a naturalistic picture: it's one thing to admit the existence of mathematical objects (arguably platonism of this sort can be, so to speak, read off of the grammar of mathematical and scientific practice). It's quite another to give them an ineliminable role in scientific explanations. (Compare the differing attitudes among scientists towards empirical and mathematical ontological posits).

(2) It seems to follow from an (independently attractive) perspective in the philosophy of mathematics, whereby (roughly) mathematical truth or objectivity is more fundamental in the order of explanation than mathematical objects.

This constraint will not be acceptable to everyone. There are those who would attempt to, so to speak, "outsource" determinacy to the mathematical objects themselves, although different outsourcing proposals differ over how reference to those objects is acheived. For instance, Sider appeals to a pretty heavy duty version of reference magnetism, while McGee appeals to the "open-endedness" of quantification. In a very different way, perhaps Gödel (on one reading) would have rejected this constraint also.

The Cognitive Constraint Human cognition cannot be attributed non-computational powers.

This constraint encodes the modern scientific view that human cognition results from neural, and hence mechanical, computations performed by the brain. Something like this is nicely expressed by McGee:

> Human beings are products of nature. They are finite systems whose behavioral responses to environmental stimuli are produced by the mechanical operation of natural forces. Thus, according to Church's Thesis, human behavior ought to be simulable by a Turing machine. This will hold even for idealized humans who never make mistakes and who are allowed unlimited time, patience, and memory.

To us, this seems conclusive. That said, the constraint is explicity rejected by some (most notably Lucas and Penrose); however, we think the arguments for their position are flawed (for the usual reasons), and it's hard to see what the alternative positive picture could be.

5 An Argument for Indeterminacy?

To put things roughly: if mathematical language is determinate, then its determinacy must arise either from the world or from our practice.

However, the two constraints just mentioned seem to put pressure on each of these options, respectively.

Accepting the metaphysical constraint seems to rule out, fairly immediately, the first.

And as for the second: according to Church's Thesis, some process is computational iff it's recursive (Turing computable, etc). But by Gödel's Incompleteness Theorem, no consistent, negation-complete theory extending a very minimal system of arithmetic is recursive. So: if the totality of our arithmetical practice is algorithmic, then there will be sentences undecided in that practice.

This can all be put together to give a *prima facie* argument for indeterminacy:

1. Mathematical determinacy arises either from the world or from our practice;
2. If the Metaphysical Constraint is true, then mathematical determinacy does not arise from the world;
3. If the Cognitive Constraint is true, then mathematical determinacy does not arise from our mathematical practice;
4. the Metaphysical Constraint is true;
5. the Cognitive Constraint is true;
6. So mathematical determinacy does not arise.

We don't endorse this argument – as it happens, we each reject it. But we do think it's the most challenging argument we're aware of for the indeterminacy of mathematics. Even if the argument is rejected, it is diagnostically valuable in pinning down the commitments of those who argue for various degrees of determinacy. To mention just a few of the salient options:

- Cosmological arguments for determinacy (Field) – rejects premise 2
- Infinitary rule-following/ rich logical resources (Carnap, Parsons, Isaacson) – rejects premise 3
- Heavy-duty platonism (Sider, McGee) – rejects premise 4
- Non-computational mental powers (Lucas, Penrose) – rejects premise 5
- Abductive/theoretical virtues approaches (Koellner, Maddy) – rejects ???
- Hyperuniverse view – rejects ???

6 Conclusion

We're not entirely sure whether there's a good argument for mathematical indeterminacy. But if there is, we think it's something like the foregoing. At the very least, proponents of determinacy need to address the metasemantic challenge, and explain how their view escapes the constraints mentioned above.