The Hyperuniverse Program: a critical appraisal

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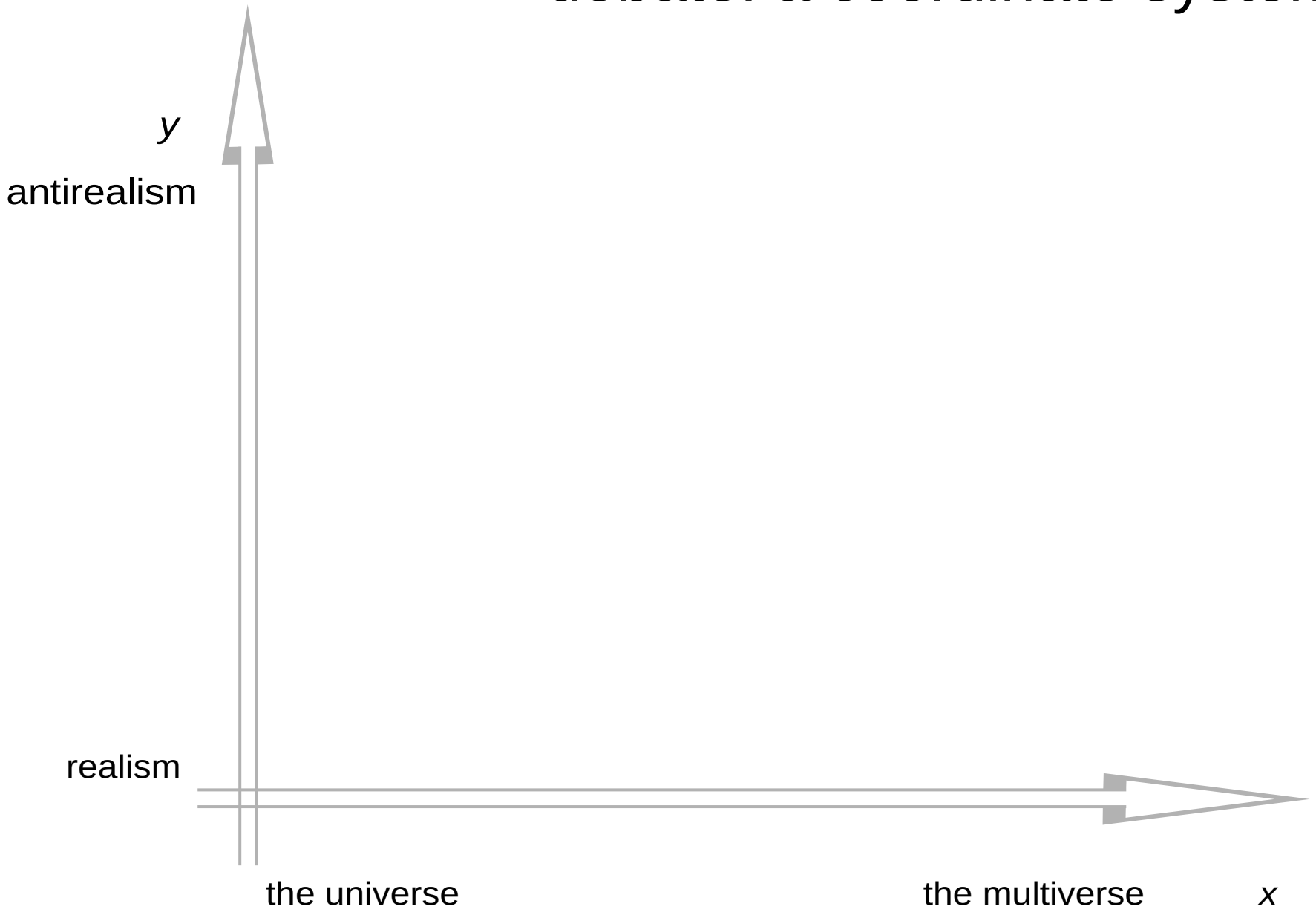
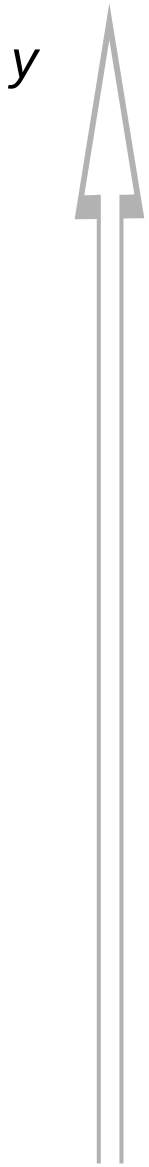
A summary

- The position of the HP in the contemporary debate in philosophy of set theory
- The HP and “the others”: analogies, differences
- The challenge to/of the HP

The position of the HP in the contemporary debate: a coordinate system



A coordinate system: the *y* axis



realism-in-truth-value: “mathematical statements have objective truth values independent of the minds, languages, conventions and such of mathematicians” (Shapiro 2005, 6)

realism-in-ontology: “at least some mathematical objects exist objectively”, i.e. mind-independently (Shapiro 2005, 6)

An aside on realism-in-truth-value: a plausible manifesto?

[…] the axioms of set theory by no means form a system closed in itself but, quite on the contrary, the very concept of set on which they are based suggest their extension by new axioms which assert the existence of still further iterations of the operation "set of" [Gödel 47-64, BP, 476]

What is of the ***concept of set***
(in a non-ontologically-realistic setting)?
Where it **comes from?**

What is of the ***concept of set***
(in a non-ontologically-realistic setting)?
Where *it* ***comes from?***

(Plausible) answer:

- ***Mathematical developments*** through which...
- Developments recognized as a *piece of* ***enlightening***, ***deep***, convincing mathematics …
- Developments leading to **"stable"** inhabitants of our mathematical world ...

Did these developments lead to a ***concept of set*** …

- really "independent of the minds, languages, conventions and such of mathematicians"?
- only regarded ***as if*** it were "independent of the minds, languages, conventions and such of mathematicians"?

The coordinate system: the *y* axis revised

Anti-realism: negation of all forms of realism

quasi-realism-in-truth-value: : mathematical statements are intentionally "**as if"** they had objective truth values independent of the minds, languages, conventions and such of mathematicians

realism-in-truth-value: "mathematical statements have objective truth values independent of the minds, languages, conventions and such of mathematicians" (Shapiro 2005, 6)

realism-in-ontology: "at least some mathematical objects exist objectively", i.e. mind-independently (Shapiro 2005, 6)

A coordinate system: the *x* axis

The-universe-view :

- *V* is a privileged structure (ontologically, epistemologically?)
- Set theory: the study of *V*
- truth in set theory: truth in *V*

The-multiverse-view:

- V is no privileged structure
- set theory: the study of the multiverse
- truth in set theory: ?

x

Miscellaneous positions in the coordinate system

realism in ontology + the-universe-view:

The set theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false. (Goedel 1947/64, in BP 476)

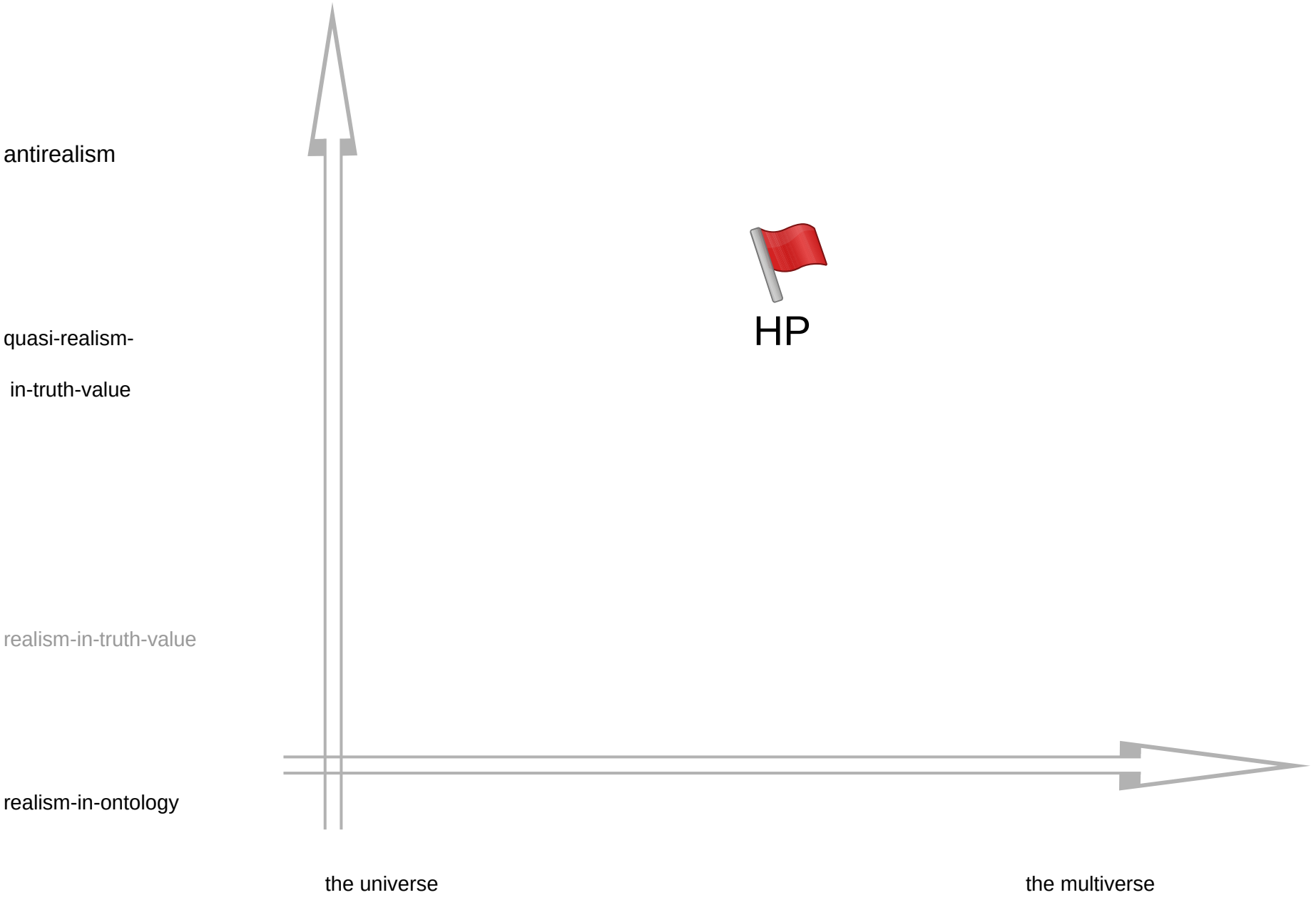
realism in ontology + the multiverse view

The multiverse view is one of higher order realism – Platonism about the universes – and I defend it as a realist position asserting actual existence of alternative set theoretic universes into which our mathematical tools have allowed us to glimpse. (Hamkins 2012, 417)

Antirealism + the multiverse view

I do not feel "a universe of ZFC" is like "the Sun"; it is rather like "a human being" or "a human being of some fixed nationality". (Shelah 2003, 211)

The position of the HP in the coordinate system



The position of the HP in the coordinate system

The HP invokes the concept of set as source of truth-values:

> […] there is possibly a hint of realism in our position, insofar as we view the concept of set as being a stable feature of our experience of sets and we subscribe to its stability in the sense that we do not question the axioms of ZFC which are true of it.

> […] realism should [extend] so far to postulate a stable concept of set, from which further properties of sets should be derived. (Friedman- Ternullo 2104, 3-5)

The HP believes in both *V* and the multiverse, endorses a **dualistic** view:

- *V* is itself "multi" (an endlessly extending in height potential tower-universe), *V* is approached through the analysis of the hyper-universe,
- but *V* is there, and the hyper-universe is described as a "technical tool allowing set theorists to use the standard model theoretic and forcing techniques so as to construe a proper context for the search of *V*".

> Within the program we are, in a sense, forced to postulate both the existence of one "extendible" universe and, at the same time, that of a plural framework containing many universes [i.e. the hyperuniverse], where properties of the universe [supposed to be satisfied by both the one "extendible" *V* and members of the hyperuniverse] allow the detection of further set theoretic truths. (Friedman-Ternullo, 2014, 10).

The novelty of the HP: ?
The HP, the *concept of set*, and Gödel

We seem to have the ability to single out the relevant concepts and properties that are derivable from it [the concept of set] (Friedman-Ternullo, 2014, 5) ...

and, more exactly,

it is possible to derive properties of the concept of set which provide us with an indication of what further properties the set theoretic hierarchy should have. (Friedman-Ternullo, 2014, 3)

Compare this with Gödel's famous quotation:

[…] the axioms of set theory by no means form a system closed in itself but, quite on the contrary, the very concept of set on which they are based suggest their extension by new axioms which assert the existence of still further iteration of the operation "set of". (Gödel 47-64, BP, 476)

The novelty of the HP: ?
The HP, the concept of set, and Gödel

Analogies: 

- ***Concept of set***: implies properties of the universe
- Properties of the universe implied by the concept of set: ***maximality properties*** (universe resulting by a maximal application of the operation “set of”, maximal extendibility in height of the universe)

Where the analogies have an end: 

- *Realism*: conceptual realism for Gödel, at best quasi-realism-in-truth-value for the HP
- The concept of set plays a more limited role in the HP, and partly an indirect, extrinsic role!

The concept of set plays a limited, indirect role in the HP (in spite of...)!

In the HP maximality is spelled out

- In terms of **existential principles** concerning **sets in** *V* (*V* as a set structure) within a potential setting (vertical maximality of *V* as indefinitely vertically extensible universe)

- in terms of the **properties of *V*** as a **model** for the axiomatic system **ZFC** in terms of width maximality principles like the IMH (if for every first-order sentence ψ, ψ is satisfied in some outer model *W* of *V*, then there is a definable inner model $V' \subseteq V$ satisfying ψ")

Both forms of maximality are supposed to be linked to the concept of set, but … in moving from sets and *V* (as a set structure) to *V* as a model a **conceptual shift** takes place! **Sets (structures of sets) ≠ Models!**

The concept of set and *V* as a model

NO:

Concept of set → *V* should be a maximal set

structure → *V* should maximize consistency

YES:

Maximality (intranscendibility)

(Maximal concept of set)

V as set structure

V as a ZFC-model

V as model and the concept of set

Concept of set

maximality of *V*
in height and width

maximality of *V*
as a model

What its plausibility is grounded on
(if not on intrinisc evidence)?

The HP, the concept of set, and ***thin realism***

Thin realist:

- has the goal of extending the realm of set theoretic truths in a quasi-realism-in-truth-value setting; in order to get new truths he/she does not need … to "posit an objective reality that the ordinary methods of set theory including classical logic do (or do not) allow us to track". For him/her there is "no such gap between the methods of set theory and sets: sets just are the sort of things that can be known about in these ways" (Maddy 2005, 362).

- Is motivated by the goal of producing a ***unified theory*** of sets serving foundational purposes

- relies on ***extrinsic evidence*** as a ***criterion of truth*** (true have to be regarded axioms that "best served existing mathematical needs and proved able to track a mathematically rich vein within the indiscriminate network of logical possibilities" (Maddy 2011, 80 ff) → regards **ZFC + large cardinal axioms** (up to a super-compact?) as a true axiomatic system for set theory.

The HP, the concept of set, and ***thin realism***

The supporter of the HP (although ready to share the quasi-realistic-in-truth-value background of the thin realist):

- does not as a primary goal/motivation that of producing a unified theory of set for foundational purposes;
- does not start from the equivalence "**extrinsic evidence (success)** ↔ **truth**";

but

- sees the **pursuit of truth** in contemporary set theory as an ***<u>end in se</u>***
- indeed he/she does reject the equivalence "**extrinsic evidence (success)** ↔ **truth** "

The HP, the concept of set, and ***thin realism***

The supporter of the HP (although perhaps ready to share the quasi-realistic-in-truth-value background of the thin realist):

- Sees *the* ***pursuit of truth*** as *a* ***goal in se***:
  ...he/she is "motivated by the desire to come as close as I can to "the right picture of V" (informal communication)

- does reject the equivalence "**extrinsic evidence (success)** ↔ **truth** ":
  (i) there may be successful mathematics that does not deserve being regarded true

  Perhaps we should accept the fact that set-theoretic truth and set-theoretic practice are quite independent of each other and not worry when we see conflicts between them. Maybe the existence of measurable cardinals is not "true" but set theory can proceed (informal communication)

- (ii) a means-end view of success may be biased, since "the development of set theory as a branch of mathematics is so rich that there will never be a consensus about which first-order axioms (beyond ZFC plus small large cardinals) best serve this development" (Ternullo-Friedman 2014, 2)

The HP: what now?

The challenge to the HP, how to:

a) ground the plausibility of the IMH and related principles as selection principles;

b) explain the *quest for set theoretic truth in V as an end-in-se,* in a quasi-realistic-in-truth-value context where:

- the concept of set plays a limited role;
- the equivalence "**extrinsic evidence (success)** $\leftrightarrow$ **truth** " is regarded as doubtful.

Suggestions?!?

A pragmatist view of truth?

there is no static or definitive aspect to [...] truth. It is a process of dynamic investigation and discovery, because neither the motivating philosophical principles (maximality, omniscience, internal unreachability, ...) nor the choice of [...] criteria instantiating them is fixed; they are subject to enrichment and improvement with the belief that as things progress one is converging towards a coherent and well-justified interpretation of set-theoretic truth. (Friedman, informal communication)

Different minds may set out with the most antagonistic views, but the progress of investigation carries them by a force outside of themselves to one and the same conclusion. This activity of thought by which we are carried, not where we wish, but to a foreordained goal, is like the operation of destiny. No modification of the point of view taken, no selection of other facts for study, no natural bent of mind even, can enable a man to escape the predestinate opinion. This great law is embodied in the conception of truth and reality. The opinion which is fated to be ultimately agreed to by all who investigate, is what we mean by the truth, and the object represented in this opinion is the real. That is the way I would explain reality. (Peirce 1878, CP 5.407).