Forcing, multiverse and realism

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This talk

In this talk we will try to understand what forcing method tells us about some philosophical aspects of set theory.

The general problems discussed are ontological (to what extend forcing sustains realism) and methodological (on the central role that forcing plays in set theoretical practice).

We will focus our attention on the modern formulation of the question about pluralism in set theory: what philosophical moral is it possible to draw from the study of a multiverse?

Multiverse positions

A multiverse is a collection of universes of set theory.

Each position takes a stand towards realism, centrality of methods allowed in set theory, truth, relativism and the aim of the study of the collection of the relevant structures: descriptive of prescriptive.

- Friedman at al.
- Hamkins et al.
- Magidor
- Shelah
- Steel
- Väänänen
- Woodin

Conceptualism

The hyperuniverse consists of the collection of all transitive countable models of ZFC.

Friedman and Ternullo. *The search for new axioms in the hyperuniverse programme*

The programme's position is that axioms do not reflect truth in an independent realm of mathematical entities. It is rather the concept of set that plays a key role in our foundational project. Now, we believe that it is possible to derive properties of the concept of set which provide us with an indication of what further properties the set-theoretic hierarchy should have.

Platonism

Hamkins *The set-theoretic multiverse*

The background idea of the multiverse, of course, is that there should be a large collection of universes, each a model of (some kind of) set theory. There seems to be no reason to restrict inclusion only to ZFC models, as we can include models of weaker theories ZF, ZF^-, KP and so on, perhaps even down to second order number theory, as this is set-theoretic in a sense. [...] On the multiverse view [...] the continuum hypothesis is a settled question; it is incorrect to describe the CH as an open problem. The answer to CH consists of the expansive, detailed knowledge set theorists have gained about the extent to which it holds and fails in the multiverse, about how to achieve it or its negation in combination with other diverse set-theoretic properties.

Forcing

"That all our knowledge begins with experience there can be no doubt."
Let us start from mathematical practice.

Forcing is a tool that allows to define new objects that sit outside a given countable transitive models (c.t.m.) of ZFC. Our main focus will be, consequently, the following questions.

Question: is there an implicit form of realism in the use of forcing?

Question: is it possible to find new methods for building new objects outside a given c.t.m.?

Cantor and Cohen

Cantor

Suppose the existence of a surjection $f : \omega \to \mathcal{P}(\omega)$. Then consider the set $A = \{n \in \omega : n \notin f(n)\}$. By surjectivity of f there is an a such that $A = f(a)$. But then $a \in A \iff a \notin f(a) \iff a \notin A$. Hence f does not exist.

Cohen

Let M be a c.t.m. of ZFC and $\mathbb{P} \in M$ a separative poset. Suppose the existence of an M-generic filter $G \subseteq \mathbb{P}$ in M. Then consider the set $E = \{p \in \mathbb{P} : p \notin G\}$. By density of E, our assumption that $G \in M$ and the genericity of G, there is an e such that $e \in E \cap G$. But then $e \in E \iff e \notin G \iff e \notin E$. Hence G does not exist in M.

An anti-realist interpretation

Observe that among the many generic filters that forcing produces there are surjections between ω and uncountable cardinals.

> **Meadows** *Naïve infinitism: the case for an inconsistency approach to infinite collections*
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> [. . .] [W]e may imagine ourselves as, so to speaks, navigating an endless collection of these countable models [. . .]. While the *illusion of multiple infinite cardinalities* is witnessed *inside* each of the universes, we are free to move between them. [. . .] I would like to make the provocative suggestion that forcing is a kind of natural revenge or dual to Cantor's theorem: where Cantor gives us the transfinite, forcing tears it down.

However, it is important not to conflate the theoretical and the meta-theoretical levels.

A realist interpretation

We start with a meta-theoretical observation.

Observation

The possibility to apply forcing to countable models tells us that the universe of all sets is not countable.

Similarly to Cantor's theorem for the theoretical level, Cohen's theorem may be considered a meta-theoretical argument for the existence of uncountable collections.

Consequently, it may be argued that also forcing (specifically Cohen's theorem on the existence of filters that are generic w.r.t. a countable model) has an existential import.

What do we do when we force?

Practically, the existential import of forcing can be seen in two ways:

1. given a c.t.m. M of ZFC, there are objects in V and not in M that we can show to exists by means of a forcing construction,

From Kunen's *Set theory*

People living in M [a c.t.m.] cannot construct a G which is $\mathbb{P}$-generic over M. They may believe on faith that there exists a being to whom their universe, M, is countable. Such a being will have a generic G and $f_G = \bigcup G$.

2. we force over V expanding the universe and we produce new sets.

From Jech's *Set theory*

The modern approach to forcing is to let the ground model be the universe V, and pretend that V has generic extensions.

What kind of existence?

What kind of realism is associated to the above types of existence?

The realism associated to the use of a c.t.m. may be called *trans-model realism*. The related notion of existence is prior to the existence in a model. Indeed, it is the existence of a generic filter that allows the construction of a generic extension. The universal class is thus existent in some sense that may be both potential or actual.

The realsim associated to the use of an extension of V may be called *trans-universe realism*. It may be thought as an abuse-of-notation version of the former. However it suggests the existence of different and independent universes of sets. So, it puts into question the foundational role of set theory.

Existential questions

There are two question that arise from considering these two forms of realism.

Question: is trans-model realism compatible with the notion of existence of sets that we find in set theory?

Question: is trans-universe realism formally treatable inside set theory?

The naturalist approach

I believe that the answer to the question about the compatibility between trans-universe realism and mathematical practice is positive, thanks to the following result.

Theorem (Hamkins)

For any forcing notion $\mathbb{P}$, there is an elementary embedding

$$V \precsim \bar{V} \subseteq \bar{V}[G]$$

of the universe V into a class model $\bar{V}$ for which there is a $\bar{V}$-generic filter $G \subseteq \bar{\mathbb{P}}$ (where $\bar{\mathbb{P}}$ is the image of $\mathbb{P}$ under the elementary embedding from V to $\bar{V}$). In particular, $\bar{V}[G]$ is a forcing extension of $\bar{V}$, and the entire extension $\bar{V}[G]$, including the embedding of V into $\bar{V}$, are definable classes in V, and $G \in V$.

The concept of set

Which concept? The *iterative conception*: sets are built in stages with the operation "set of".

The iterative conception originates from Zermelo's *cumulative hierachy* structure (and Gödel's L) as the intended interpretation of ZFC.

The quasi-categoricity character of the cumulative hierarchy of sets has the consequence of thinning the difference between domains of sets and sets.

Zermelo *On boundary numbers and domains of sets* 1930

[E]very categorically determined domain can also be conceived of as a 'set' in some way or another.

The existence of sets in V is prior to their existence in first order models.

Quasi-combinatorialim

This notion of existence prior to existence in a domain is connected to a philosophical clarification of the notion of *arbitrary* set.

Bernays *On mathematical platonism* 1935

But analysis is not content with this modest variety of platonism [to take the collection of all numbers as given]; it reflects it to a stronger degree with respect to the following notions: set of numbers, sequence of numbers, and function. It abstracts from the possibility of giving definitions of sets, sequences, and functions. These notion are used in a 'quasi-combinatorial' sense, by which I mean: in the sense of an analogy of the infinite to the finite.
From this point of view, constructive definitions of specific functions, sequences, and sets are only ways to pick out an object which exists independently of, and prior to, the construction. The axiom of choice is an immediate application of the quasi-combinatorial concepts in question.

Genericity and arbitrarines

So far I suggested an analogy between the realist component implicit in the use of forcing and the widespread presence of the notion of quasi-combinatorialism in mathematical practice.

Suggestion

Forcing is a tool to define mathematical objects that exist prior and independently of the means given by a countable model.

What I propose is a step forward in this analogy that connects arbitrary sets and generic sets.

Absolute arbitrary sets?

Exactly like the notion of arbitrary set, that of generic set presupposes a difficult question.

Question

Is the notion of genericity absolute of relative to a given model of ZFC?

Given a c.t.m. M and a set X such that $\mathcal{P}(X)$ is not contained in M we can divide $\mathcal{P}(X)$ in $Def_M(X)$: the subsets of X definable in M, and $Arb_M(X)$: the subsets of X arbitrary with respects to M.

Among $Arb_M(X)$ there are sets that belong to generic extension of M. But is it possible to find sets that will never be captured by forcing extensions of M? Is it possible to graduate the arbitrariness of sets by means of genericity?

Mostowski suggestion

There is an antecedent of this proposal in Mostowski's discussion of the philosophical implication of the independence results.

Mostowski *Recent results in set theory* 1967

Models constructed by Gödel and Cohen are important not only for the purely formal reasons that they enable us to obtain independence proofs, but also because they show us various possibilities which are open to us when we want to make more precise the intuitions underlying the notion of a set. Owing to Gödel's work we have a perfectly clear intuition of a set which is predicatively defined by means of a transfinite predicative process. No such clear interpretation has as yet emerged from Cohen's models because we possess as yet no intuition of generic sets; we only understand the relative notion of a set which is generic with respect to a given model.

A forcing of models of ZFC

Definition (the Γ set-generic multiverse of M)

Given a c.t.m. M and a class of forcings Γ, we define $\mathbb{M}^{\Gamma}_{M}$ to be the set consisting of all generic extensions of M given by elements of Γ. Moreover for $N, N' \in \mathbb{M}^{\Gamma}_{M}$ we set $N' \leq_{\mathbb{M}^{\Gamma}_{M}} N$, whenever N' is a generic extension of N by means of a notion of forcing $\mathbb{P} \in \Gamma \cap N$.

1) The relation $\leq_{\mathbb{M}^{\Gamma}_{M}}$ is antisymmetric by definition of inclusion.

2) If the trivial forcing is included in Γ, then $\leq_{\mathbb{M}^{\Gamma}_{M}}$ is reflexive.

3) If Γ is closed under two step iteration, then $\leq_{\mathbb{M}^{\Gamma}_{M}}$ is transitive.

If the above conditions hold, then $(\mathbb{M}^{\Gamma}_{M}, \leq_{\mathbb{M}^{\Gamma}_{M}})$ is a poset.

First example

Let Γ consists of Cohen forcing (together with the trivial one). Fix a c.t.m. M and define $\mathbb{M}_M^C$ to be the poset of the Cohen-generic multiverse of M.

Fact (Woodin)

If M is a countable model of ZFC set theory, then there are forcing extensions $M[c]$ and $M[d]$, both obtained by adding a Cohen real, which are non-amalgamable in the sense that there can be no model of ZFC with the same ordinals as M containing both $M[c]$ and $M[d]$.

Corollary

If we force with $\mathbb{M}_M^C$ over a c.t.m W we obtain a proper generic extension of W.

Properties of $\mathbb{M}_M^C$

Theorem (Hamkins, V.)

Let M be a countable transitive model of ZFC and let $\langle c_n : n \in \omega \rangle$ be a tower of (finitely) mutually M-generic Cohen reals

$$M \subseteq M[c_0] \subseteq M[c_0][c_1] \subseteq \dots$$

Then there is a M-generic Cohen real d such that $c_n \in M[d]$. As a consequence $M[c_0][c_1] \dots [c_n] \subseteq M[d]$. Furthermore, d is $M[c_0, \dots c_n]$-generic.

Corollary

The forcing $\mathbb{M}_M^C$ is σ-closed and so it preserves ω_1.

More genericity

Question: what about forcing with $\mathbb{M}^{\Gamma}_{M}$, where Γ consists of all possible notions? Let us call this forcing $\mathbb{M}_{M}$.

Fact

If we force with $\mathbb{M}_{M}$ over a c.t.m W we obtain a proper generic extension of W.

If we force with $\mathbb{M}_{M}$ we get a set that is more generic (or arbitrary) than the ones we can get from M alone.

A generic model?

Unfortunately forcing with $\mathbb{M}_C$ does not produce a model of ZFC, since we add class-many reals (from the point of view of M). And so the powerset axiom is not preserved.

Question: what about forcing with $\mathbb{M}_M$?

Again, forcing with $\mathbb{M}_M$ does not produce a model of ZFC, because by genericity everything is collapsed to countable and so the powerset axiom does not hold. But I am now working on a modification of $\mathbb{M}_M$ intended to build a truly generic model.

Interesting related questions

Question 1

Is it possible to characterize a given class Γ by means of the properties of associated partial orders $\mathbb{M}_\Gamma(M)$, for some c.t.m. M? For which classes Γ the poset $\mathbb{M}_\Gamma(M)$ is c.c.c.? or σ-closed? or proper?

Question 2

Given a multiverse with some specific structural properties, is it possible to find a specific class of posets Γ (and a model M) such that the corresponding generic multiverse structure corresponds to $\mathbb{M}_\Gamma$?

Question 3

It is possible to extend the idea of forcing with models to other logical items (e.g. theories)?

Thank you!